

The effect of small viscosity and diffusivity on the marginal stability of stably stratified shear flows

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The effect of non-zero, but small, viscosity and diffusivity on the marginal stability of a stably stratified shear flow is examined by making perturbations around the neutral solution for an inviscid and non-diffusive flow. The results apply to turbulent flows in which horizontal and vertical turbulent transports of momentum and buoyancy are represented by eddy coefficients of viscosity and diffusivity that vary in the vertical (z) direction. General expressions are derived for the modified phase speed and the growth rate of small disturbances as a function of wavenumber. To first order in their coefficients, the effect on the phase speed of adding viscosity and diffusivity is zero. Growth rates are found for two mean flows when the horizontal or vertical coefficients of viscosity and diffusivity vary in z in such a way that the rates can be found analytically. The first flow, denoted as a ‘Holmboe flow’, has a velocity and density interface: the mean horizontal velocity and the density are both proportional to $\tanh az$, where a is proportional to the inverse of the interface thickness. The second, ‘Drazin flow’, has a similar velocity variation in z but uniform density gradient. The analytical results compare favourably with numerical calculations. Small horizontal coefficients of viscosity and diffusivity may affect disturbances to the flow in opposite ways. Although the effect of uniform vertical coefficients of viscosity is to decrease the growth rates, and uniform vertical coefficients of diffusivity increase them, cases are found in which, with suitably chosen z dependence, vertical coefficients of viscosity (or diffusivity) may cause a previously neutral disturbance to grow (or to diminish); viscosity may destabilize a stably stratified shear flow. The introduction of viscosity and diffusivity may consequently increase the critical Richardson number to a value exceeding $1/4$. While some patterns of behaviour are apparent, no simple rule appears to hold about whether flows that are neutral in the absence of these effects (viscosity or diffusivity) will be stabilized or destabilized when they are added. One such rule, namely our conjecture that viscosity is always stabilizing and that diffusivity is destabilizing, is explicitly refuted.

Key words: geophysical and geological flows, instability, stratified turbulence

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1. Introduction

Naturally occurring flows are generally turbulent with transfers of momentum and buoyancy (often in the form of heat) that are non-zero. These turbulent transfers, represented by vertical and horizontal eddy coefficients of viscosity and diffusivity, have been used by Liu, Thorpe & Smyth (2012) in the equations of motion and density conservation to examine the effect of turbulence on the stability of stably stratified flows, with application to those measured in the estuary of the River Clyde in western Scotland. In that study, equality of the vertical coefficients of viscosity, $A_V(z)$, and diffusivity, $K_V(z)$, is assumed. The coefficient $K_V(z)$, a function only of the vertically upwards coordinate, z , is determined from time-averaged available data using the Osborn (1980) formula, $K_V = \gamma e/N^2$, where e is the rate of dissipation of turbulent kinetic energy per unit mass, N is the buoyancy frequency and γ is a constant equal to ~ 0.2 . Both e and N usually vary with z , and therefore so do the two eddy coefficients. Cases are examined which also include horizontal eddy coefficients, $A_H(z)$ and $K_H(z)$. More recently, the method has provided explanations of both the diurnal mixing cycle and the deep cycle of turbulence in the upper Equatorial Pacific (Smyth *et al.* 2014). There is consequently interest in determining the general effects of viscosity and diffusion parameters that depend on z on the stability of stably stratified shear flows. Although the analysis described below will apply to the effects of molecular viscosity and diffusivity as well as to turbulence, for simplicity we adopt the commonly used symbols of the turbulent eddy coefficients rather than those of the molecular properties (e.g. we use A_V and A_H rather than ν for kinematic viscosity). How well turbulence may be represented by such eddy coefficients is not addressed here.

Our objective is to determine how, with values of wavenumber k and minimum Richardson number J fixed to be those on the neutral curve of an inviscid and non-diffusive flow, the addition of small viscosity or diffusivity affects the phase speed and growth rate of small disturbances. This involves an examination of the local structure of the neutral surface, not only in k and J space, but in a space that also includes viscosity and diffusion parameters.

Section 2 describes the perturbation of the neutral solution of a steady stratified horizontal shear flow, $U(z)$, using a small ordering parameter, δ , to characterize the magnitude of the viscosity and diffusivity coefficients. The results are applied § 3 to two particular flows.

The first, denoted for simplicity as the ‘Holmboe flow’, is

$$U(z) = U_0 \tanh az \quad \text{and} \quad N^2(z) = N_0^2 \operatorname{sech}^2 az, \quad (1.1)$$

representing the flow with an interface of thickness $\sim 2a^{-1}$ at $z = 0$, where $N^2 = -(g/\rho_0) d\rho/dz$ implies a density variation, ρ , like the horizontal velocity, proportional to $\tanh az$. The neutral curve, $J = J_c(k)$, on which small periodic disturbances of wavenumber k have zero growth rate, is given by

$$J_c = \kappa(1 - \kappa), \quad 0 \leq \kappa \leq 1, \quad (1.2)$$

where $\kappa = k/a$ is the non-dimensional wavenumber. Flows with $J_c < 1/4$, the maximum value of J_c in (1.2) found at $\kappa = 1/2$, are unstable for small disturbances within a finite band around $\kappa = 1/2$. (According to Miles 1961; Hazel 1972 the neutral curve is first given in UCLA lecture notes by Holmboe in 1960.) Smyth, Moum & Nash (2011) find that \tanh profiles of U and ρ are useful in the analysis of equatorial flows. The Holmboe flow is also used in a numerical analysis (including

non-uniform A_V and K_V) of the diurnal cycle of shear instability in the upper Equatorial Pacific by Smyth *et al.* (2014).

The second flow, the ‘Drazin flow’,

$$U(z) = U_0 \tanh az \quad \text{and} \quad N^2(z) = N_0^2, \text{ a constant,} \quad (1.3)$$

first studied by Drazin (1958), provides results that may usefully be compared to those using the ‘Holmboe flow’. The neutral curve is given by

$$J_c = \kappa^2(1 - \kappa^2), \quad 0 \leq \kappa \leq 1. \quad (1.4)$$

Flows with $J_c < 1/4$, the maximum value of J_c in (1.4) found at $\kappa^2 = 1/2$ ($\kappa \approx 0.707$), are unstable to small disturbances.

Preliminary results described by Smyth *et al.* (2014) using the numerical method of Liu *et al.* (2012) find that a non-zero but constant A_V , and zero K_V , A_H and K_H , results in a reduction in the growth rates of unstable modes as expected because of the viscous dissipation of disturbance energy. However a constant K_V with zero A_V , A_H and K_H , has the opposite effect, increasing the growth rates. A reduction of growth rates found when $Pr = A_V/K_V = 1$ (and also when $Pr = 0.72$; Maslowe & Thompson 1971) suggests that at moderate and large Pr the stabilizing effect of viscosity dominates over the destabilizing effect of non-zero diffusivity. The present authors conjectured that viscosity might always be stabilizing and diffusivity destabilizing. By examining viscosity and diffusivity coefficients that vary in z it is shown here that this conjecture is false. Growth rates of small disturbances to the Holmboe and Drazin flows are found analytically in § 3 and confirmed numerically in § 4. The examples of §§ 3 and 4 are used in § 5 to test the conjectures about the general effects of small viscosity and diffusivity, particularly how the critical Richardson number is changed.

2. Analysis

Following Liu *et al.* (2012), it is supposed that the temporal rates of change of a horizontal unidirectional mean shear flow, U , and its buoyancy frequency, N , resulting from the effects of viscosity or diffusivity, are very small compared to the rates of growth or decay of small disturbances. (The validity of this assumption is considered in § C.1.) Assuming a two-dimensional perturbation to the mean flow with stream function, ψ , and density, ρ , the Boussinesq equation for vorticity becomes

$$\begin{aligned} \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 \psi - U'' \frac{\partial \psi}{\partial x} + \mathcal{J}(\nabla^2 \psi) &= \frac{g}{\rho_0} \frac{\partial \rho}{\partial x} + A_H \frac{\partial^2}{\partial x^2} \nabla^2 \psi + A_H' \frac{\partial^3 \psi}{\partial x^2 \partial z} \\ + A_V \frac{\partial^2}{\partial z^2} \nabla^2 \psi + A_V' \left(\frac{\partial^2}{\partial x^2} + 2 \frac{\partial^2}{\partial z^2} \right) \frac{\partial \psi}{\partial z} + A_V'' \frac{\partial^2 \psi}{\partial z^2}, \end{aligned} \quad (2.1)$$

and density conservation implies

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \rho + \frac{\rho_0 N^2}{g} \frac{\partial \psi}{\partial x} + \mathcal{J}(\rho) = K_H \frac{\partial^2 \rho}{\partial x^2} + K_V \frac{\partial^2 \rho}{\partial z^2} + K_V' \frac{\partial \rho}{\partial z}, \quad (2.2)$$

where \mathcal{J} is the Jacobean operator $(\partial \psi / \partial z) \partial / \partial x - (\partial \psi / \partial x) \partial / \partial z$, g is the acceleration due to gravity and ρ_0 is the overall mean density. The prime indicates total differentiation with respect to z (e.g. $U'' \equiv d^2 U / dz^2$, $A_V' \equiv dA_V / dz$). Excepting the nonlinear (\mathcal{J}) terms and the use of density rather than buoyancy, these equations are as given by Liu *et al.* Extension to three-dimensional flows is straightforward.

It is supposed that the viscosity and diffusion coefficients are functions only of z and are of the order of a small parameter, δ , so they may be written δA_{V^*} , δK_{V^*} , δA_{H^*} and δK_{H^*} , where A_{V^*} etc. are of order one. The stream function of a spatially periodic disturbance of wavenumber k is written $\psi = \varepsilon[\phi_1(z) + \delta\phi_{11}(z)] \exp[ik(x - c_0t)] + (\text{terms of order } \varepsilon^2 \text{ and } \delta\varepsilon^2 \text{ or higher-order})$, and likewise density, ρ , in terms of $\rho_1(z)$ and $\rho_{11}(z)$, where ε is an ordering parameter corresponding to a perturbation of the inviscid and non-diffusive flow, and the phase speed is $c_0 = (c_{0r} + ic_{0i}) + \delta(c_{1r} + ic_{1i})$. The functions, ϕ_1 , ϕ_{11} , ρ_1 and ρ_{11} , are taken to be real. The lowest-order (ε^1) phase speed of a perturbation, that of an inviscid and non-diffusive flow, is c_{0r} , and the growth rate is kc_{0i} . The lowest-order phase speed of a perturbation of the viscous and diffusive flow is $c_{0r} + \delta c_{1r}$, and the growth rate is $kc_{0i} + \delta kc_{1i}$. In examining the effect of viscosity and diffusivity on a flow that, in the absence of viscosity and diffusivity, is neutral, the lowest-order growth rate, kc_{0i} , is zero, and the growth rate of a perturbation is δkc_{1i} . We seek to determine this growth rate by substituting for ψ and ρ in (2.1) and (2.2) and comparing terms of orders ε and $\delta\varepsilon$.

Retaining only terms of order ε , writing $W = U - (c_{0r} + ic_{0i})t$, and comparing coefficients of $\exp[ik\{x - (c_{0r} + ic_{0i})t\}]$, (2.1) leads to

$$W \left(\frac{d^2}{dz^2} - k^2 \right) \phi_1 - \phi_1 U'' - g \frac{\rho_1}{\rho_0} = 0 \tag{2.3}$$

and (2.2) gives

$$W\rho_1 + \frac{\rho_0 N^2}{g} \phi_1 = 0. \tag{2.4}$$

From (2.4), $\rho_1 = -(\rho_0 N^2/g)\phi_1/W$ which, substituted into (2.3), gives the equation to order ε for the periodic disturbance to the mean flow:

$$\mathcal{L}(\phi_1) \equiv \left[\frac{d^2}{dz^2} - k^2 - \frac{U''}{W} + \frac{N^2}{W^2} \right] \phi_1 = 0. \tag{2.5}$$

This is the well-known Taylor–Goldstein equation of inviscid and non-diffusive stratified shear flow (Thorpe 1969; Drazin & Reid 1981), solutions of which are exponentially growing and therefore unstable if $kc_{0i} > 0$.

To examine the effects of viscosity and diffusivity, we retain terms of order $\varepsilon\delta$ after substituting ψ , ρ and $A_V = \delta A_{V^*}$, etc. into (2.1) and (2.2). Eliminating ρ_1 as before we find

$$\begin{aligned} & ikW^2 \mathcal{L}(\phi_{11}) + k(c_{1i} - ic_{1r}) \left[W \left(\frac{d^2}{dz^2} - k^2 \right) \phi_1 - \frac{N^2 \phi_1}{W} \right] \\ &= WA_{V^*} \left(\frac{d^2}{dz^2} - k^2 \right) \phi_1'' + WA_{V^*}' \left(2 \frac{d^2}{dz^2} - k^2 \right) \phi_1' + WA_{V^*}'' \phi_1 \\ &\quad - K_{V^*} \left(\frac{N^2 \phi_1}{W} \right)'' - K_{V^*}' \left(\frac{N^2 \phi_1}{W} \right)' \\ &\quad - A_{H^*} W k^2 \left(\frac{d^2}{dz^2} - k^2 \right) \phi_1 - W k^2 A_{H^*}' \phi_1' + k^2 K_{H^*} \frac{N^2 \phi_1}{W}. \end{aligned} \tag{2.6}$$

If ϕ_1 and ϕ_{11} are zero at the upper and lower boundaries of the flow (making the vertical velocity there zero) then ϕ_1 and ϕ_{11} are adjoint in the sense that $\int \phi_{11} \mathcal{L}(\phi_1) dz = \int \phi_1 \mathcal{L}(\phi_{11}) dz$, where the integrals are taken over the depth of

the flow. Hence using (2.5):

$$\int \phi_1 \mathcal{L}(\phi_{1i}) dz = 0. \tag{2.7}$$

Multiplying (2.6) by ϕ_1/W^2 , integrating over the range of z , and using (2.7), we obtain an expression for the change in the wave speed ($c_{1r} + ic_{1i}$) that results from the presence of viscosity and diffusivity with small coefficients, and their derivatives. On the inviscid and non-diffusive neutral curve, $c_{0i} = 0$ and so $W = (U - c_{0r})$ is real. It follows from the imaginary part of the integrated equation that

$$c_{1r} = 0, \tag{2.8}$$

i.e. the phase speed on the neutral curve is not affected by viscosity or diffusivity at this order, whatever their variation with z . The real part of the integrated equation gives the growth rate:

$$\begin{aligned} kc_{1i} = & \left\{ \int A_{V^*} \frac{\phi_1}{W} \left(\frac{d^2}{dz^2} - k^2 \right) \phi_1'' dz + \int A'_{V^*} \frac{\phi_1}{W} \left(2 \frac{d^2}{dz^2} - k^2 \right) \phi_1' dz + \int A''_{V^*} \frac{\phi_1}{W} \phi_1'' dz \right. \\ & - \int K_{V^*} \frac{\phi_1}{W^2} \left(\frac{N^2 \phi_1}{W} \right)'' dz - \int K'_{V^*} \frac{\phi_1}{W^2} \left(\frac{N^2 \phi_1}{W} \right)' dz \\ & - k^2 \int A_{H^*} \frac{\phi_1}{W} \left(\frac{d^2}{dz^2} - k^2 \right) \phi_1 dz - k^2 \int A'_{H^*} \frac{\phi_1}{W} \phi_1' dz \\ & \left. + k^2 \int K_{H^*} \frac{N^2 \phi_1^2}{W^3} dz \right\} / I, \quad \text{where } I = \int \frac{\phi_1^2}{W} \left(\frac{U''}{W} - 2 \frac{N^2}{W^2} \right) dz. \tag{2.9} \end{aligned}$$

(The integral I may be recognized as the same denominator as found by Howard 1963 in his examination of neutral curves to which we refer again in § 5.3.) The expression is to be evaluated on the neutral curve, the solution of the Taylor–Goldstein equation (2.5) when $c_{0i} = 0$ and $J = J_c(k)$.

3. Examples using chosen flows

3.1. The Holmboe flow

Analytical expressions for the neutral curves and for the neutral eigenfunction, ϕ_1 , are available for certain $U(z)$ and $N^2(z)$ profiles (e.g. Drazin & Howard 1966; Thorpe 1969; Drazin & Reid 1981) and, in principle, these can be substituted into (2.9) to find analytical expressions for kc_{1i} .

We first consider the Holmboe flow (1.1) in a fluid of infinite depth. The neutral eigenfunction is

$$\phi_1 = A |\sinh az|^{(1-\kappa)} \operatorname{sech} az, \tag{3.1}$$

with amplitude A , and with $c_{0r} = 0$ and neutral curve given by (1.2). When appropriate choices are made for the horizontal and vertical eddy coefficients, e.g. those listed in table 1, integrals in (2.9) can be found using formulae devised by Howard (1963) that are summarized in appendix A.

Since there are no products of the coefficients A_{V^*} , K_{V^*} , etc. in (2.9) (i.e. they are not interactive at order $\varepsilon\delta$) the effect on the growth rate of taking two or more of the coefficients to be non-zero is found by a summation. In general the growth rates of disturbances are of the form

$$\delta kc_{1i} = a^2 [A_{V0} \kappa f_{AV}(\kappa) + K_{V0} \kappa f_{KV}(\kappa) + A_{H0} \kappa f_{AH}(\kappa) + K_{H0} \kappa f_{KH}(\kappa)], \tag{3.2}$$

Eddy coefficient	$\kappa f_{AV}(\kappa)$ etc.	Effect on growth rate/stability	$\kappa f_{AV}(\kappa)$ at $\kappa = 0.5$	
			Theory	Numerical
Vertical				
A_{V0} (constant)	Not found	Not found	Not found	-3.7479
K_{V0} (constant)	Not found	Not found	Not found	1.6938
$A_{V0}\tanh^2 az$	κ	(>0): destab.	0.5	0.5220
$K_{V0}\tanh^2 az$	$-\kappa(1 - \kappa)/2$	(<0): stab.	-0.1250	-0.1214
$A_{V0}\text{sech}^2 az$	Not found	Not found	Not found	-4.2834
$K_{V0}\text{sech}^2 az$	Not found	Not found	Not found	1.8209
$A_{V0}\tanh^4 az$	$(1 - \kappa)(2 + 11\kappa - 3\kappa^2)/12$	(>0): destab.	0.2813	0.2861
$K_{V0}\tanh^4 az$	$-(1 - \kappa)(2 + 9\kappa + \kappa^2)/12$	(<0): stab.	-0.2813	-0.2826
$A_{V0}\tanh^2 az \text{sech}^2 az$	$-(2 - 3\kappa - 14\kappa^2 + 3\kappa^3)/12$	(<0): stab.	0.2188	0.2394
		if $0 < \kappa < 0.293$ or		
		(>0): destab.		
		if $0.293 < \kappa \leq 1$		
		(>0): destab.	0.1563	0.1591
$K_{V0}\tanh^2 az \text{sech}^2 az$	$(2 + \kappa)(1 - \kappa^2)/12$			
Horizontal				
A_{H0} (constant)	$-\kappa(1 + \kappa)/2$	(<0): stab.	-0.3750	-0.3703
K_{H0} (constant)	$\kappa(1 - \kappa)/2$	(>0): destab.	0.1250	0.1203
$A_{H0}\tanh^2 az$	$-\kappa^2(1 - \kappa)/2$	(<0): stab.	-0.0625	-0.0624
$K_{H0}\tanh^2 az$	$-\kappa^2(1 - \kappa)/2$	(<0): stab.	-0.0625	-0.0616
$A_{H0}\text{sech}^2 az$	$-\kappa(1 + \kappa^2)/2$	(<0): stab.	-0.3125	-0.3131
$K_{H0}\text{sech}^2 az$	$\kappa(1 - \kappa^2)/2$	(>0): destab.	0.1875	0.1886
$A_{H0}\tanh^2 az \text{sech}^2 az$	$\kappa^2(1 - \kappa)^2/4$	(>0): destab.	0.0156	0.0153
$K_{H0}\tanh^2 az \text{sech}^2 az$	$-\kappa^2(1 - \kappa^2)/4$	(<0): stab.	-0.0469	-0.0463

TABLE 1. The Holmboe flow. Expressions for the growth rate terms, $\kappa f_{AV}(\kappa)$, $\kappa f_{KV}(\kappa)$, etc. appearing in (3.2) and (3.3), etc. for viscosity and diffusivity that vary with z , showing the effect of viscosity and diffusivity on the stability of small disturbances that, in an inviscid and non-diffusive tanh flow given by (1.1), are neutrally stable. (The non-dimensional wavenumber, $\kappa = k/a$, is >0.) ‘stab.’ denotes that the viscous or diffusive effects are stabilizing and ‘destab.’ those that are destabilizing. Values of $\kappa f_{AV}(\kappa)$ etc. at $\kappa = 0.5$ are calculated using the numerical method (§4 and appendix B) and, when solutions can be found, from the theoretical analysis (§3). Where theoretical values are ‘not found’ this is because some of the integrals in (2.9) are singular.

where the coefficients, A_{V0} , K_{V0} etc. characterize the magnitude of the vertical and horizontal coefficients of viscosity and diffusivity. The non-dimensional growth rates, $\sigma = \delta kc_{1i}/aU_0$, may be written

$$\sigma = R_V^{-1}[\kappa f_{AV}(\kappa) + Pr_V^{-1}\kappa f_{KV}(\kappa)] + R_V^{-1}[\kappa f_{AH}(\kappa) + Pr_V^{-1}\kappa f_{KH}(\kappa)], \quad (3.3)$$

where $R_V = U_0/aA_{V0}$ and $R_H = U_0/aA_{H0}$ are vertical and horizontal Reynolds numbers, and $Pr_V = A_{V0}/K_{V0}$ and $Pr_H = A_{H0}/K_{H0}$ are vertical and horizontal Prandtl numbers, respectively.

For example, when $A_V = K_V = 0$, and A_H and K_H are both constant (denoted as A_{H0} and K_{H0} , respectively, in table 1), (3.3) becomes

$$\sigma = R_H^{-1}[\kappa f_{AH}(\kappa) + Pr_H^{-1}\kappa f_{KH}(\kappa)], \quad (3.4)$$

where $\kappa f_{AH}(\kappa) = -\kappa(1 + \kappa)/2$ and $\kappa f_{KH}(\kappa) = \kappa(1 - \kappa)/2$, respectively, as given in table 1. Because $\kappa f_{AH}(\kappa)$ is negative in the range $0 \leq \kappa \leq 1$ of the neutral curve, small A_H acts to decrease the growth rates from their zero value on the inviscid and non-diffusive neutral curve; the effect of a vertically uniform ‘horizontal viscosity’ (i.e. adding the effect of a horizontal coefficient of viscosity representing horizontal transfers of momentum by viscosity or turbulent eddies) is to make the growth rates negative or to stabilize the previously neutral flow. In contrast the effect of a uniform ‘horizontal diffusivity’, with $\kappa f_{KH}(\kappa) > 0$, is to destabilize the flow. The sign of the net growth rate in (3.3) depends on the relative sizes of A_H and K_H . For example, when the coefficients, A_H and K_H , are taken to be not only constant, as above, but equal so that $Pr_V = 1$, the sum the two coefficients in the expression for σ , equal to $-\kappa^2$, is negative, implying a change in growth rate that is <0 , and the net effect is to stabilize the flow.

Since $\text{sech}^2 az = 1 - \tanh^2 az$, the expression for $\kappa f_{KH}(\kappa)$ when $K_H = K_{H0}\text{sech}^2 az$ is found by subtracting those of $K_H = \text{constant}$ and $K_H = K_{H0}\tanh^2 az$, and similarly for $A_H = A_{H0}\text{sech}^2 az$. Similar conclusions for A_V and K_V follow using $\tanh^2 az - \tanh^2 az \text{sech}^2 az = \tanh^4 az$, and are consistent with table 1.

3.2. The Drazin flow

To examine the sensitivity of the results to the ambient stratification, we choose as a second example the ‘Drazin flow’ (1.3). This differs from the Holmboe flow in that the density gradient is constant, with no interfacial layer. The neutral eigensolution is

$$\phi_1 = A|\sinh az|^{(1-\kappa^2)} \text{sech} az, \quad (3.5)$$

and the neutral curve is given by (1.4). The critical Richardson number is $1/4$, found when $\kappa^2 = 1/2$. Howard (1963) shows that the flow is stable if $J > J_c$, and unstable otherwise. The integrals in (2.9), leading to growth rates represented by $\kappa f_{AV}(\kappa)$ etc. can be found for coefficients given in table 2.

4. The numerical calculations

Numerical methods described in appendix B are used to test the analytical solutions for modes near the critical limiting value of the inviscid and non-diffusive stability boundary at $J_c = 1/4$ and $\kappa = 1/2$ for the Holmboe flow or $\kappa^2 = 1/2$ for the Drazin flow. The numerical calculation yields values of $\kappa f_{AV}(\kappa)$ etc. in (3.2) and (3.3) at the limiting wavenumbers. Values are given in tables 1 and 2. The agreement of theoretical and numerical values is generally good; the average of the modulus of the difference between the values is less than 2.7% of the average of the modulus of

Eddy coefficient	$\kappa f_{AV}(\kappa)$ etc.	Effect on growth rate/stability	$\kappa f_{AV}(\kappa)$ at $\kappa = 0.5$	
			Theory	Numerical
Vertical				
A_{V0} (constant)	Not found	Not found	Not found	-1.999
K_{V0} (constant)	Not found	Not found	Not found	-0.0043*
$A_{V0} \tanh^2 az$	κ^4	(>0): destab.	0.25	0.2492
$K_{V0} \tanh^2 az$	$\kappa^2(1 - \kappa^2)/2$	(>0): destab.	0.125	0.1037*
$A_{V0} \operatorname{sech}^2 az$	Not found	Not found	Not found	-2.1772
$K_{V0} \operatorname{sech}^2 az$	Not found	Not found	Not found	-0.1271*
$A_{V0} \tanh^4 az$	$\kappa^2(1 - \kappa^2)(1 + 4\kappa^2)/6$	(>0): destab.	0.125	0.1300
$K_{V0} \tanh^4 az$	$-\kappa^4(1 - \kappa^2)$	(<0): stab.	-0.125	-0.1365
$A_{V0} \tanh^2 az \operatorname{sech}^2 az$	$-\kappa^2(1 + \kappa^2)(1 - 4\kappa^2)/6$	(<0): stab. if $\kappa < 1/2$ (>0): destab. if $\kappa > 1/2$	0.125	0.1195
$K_{V0} \tanh^2 az \operatorname{sech}^2 az$	$\kappa^2(1 - \kappa^2)(1 + 2\kappa^2)/2$	(>0): destab.	0.25	0.2417
Horizontal				
A_{H0} (constant)	$-\kappa^2$	(<0): stab.	-0.5	-0.5000
K_{H0} (constant)	0	(=0): neutral	0	0.0071*
$A_{H0} \tanh^2 az$	$-\kappa^2(1 - \kappa^2)/2$	(<0): stab.	-0.125	-0.125
$K_{H0} \tanh^2 az$	$-\kappa^2(1 - \kappa^2)/2$	(<0): stab.	-0.125	-0.1196
$A_{H0} \operatorname{sech}^2 az$	$-\kappa^2(1 + \kappa^2)/2$	(<0): stab.	-0.375	-0.3751
$K_{H0} \operatorname{sech}^2 az$	$\kappa^2(1 - \kappa^2)/2$	(>0): destab.	0.125	0.1267
$A_{H0} \tanh^2 az \operatorname{sech}^2 az$	0	(=0): neutral	0	-0.0002*
$K_{H0} \tanh^2 az \operatorname{sech}^2 az$	$-\kappa^4(1 - \kappa^2)/2$	(<0): stab.	-0.063	-0.0643

TABLE 2. The Drazin flow. Expressions for the growth rate terms, $\kappa f_{AV}(\kappa)$, $\kappa f_{KV}(\kappa)$, etc. appearing in (3.2) and (3.3), etc. for viscosity and diffusivity that vary with z , showing the effect of viscosity and diffusivity on the stability of small disturbances that, in an inviscid and non-diffusive flow, $U(z) \sim \tanh az$, with uniform buoyancy frequency, N , are neutrally stable. (The non-dimensional wavenumber, $\kappa = k/a$, is >0.) 'stab.' denotes that the viscous or diffusive effects are stabilizing and 'destab.' those that are destabilizing. 'neutral' implies that the z -variations of viscosity or diffusivity have no effect, i.e. the growth rates, $\kappa f_{AV}(\kappa)$, $\kappa f_{KV}(\kappa)$, etc. are zero. The asterisk, *, denotes where the stabilizing or destabilizing effects of viscosity or diffusivity differ from those of the Holmboe flow in table 1. Values of $\kappa f_{AV}(\kappa)$ etc. at $\kappa = 0.707$ (i.e. $\kappa^2 = 1/2$) are calculated using the numerical method (§ 4 and appendix B) and, when solutions can be found, from the theoretical analysis (§ 3). Where theoretical values are 'not found' this is because some of the integrals in (2.9) are singular.

the theoretical values. The small inaccuracies appear to originate from the numerical estimates and not from the theory. The numerical method is used to compute $\kappa f_{AV}(\kappa)$ etc. for some cases in which integrals in (2.9) are singular at $z = 0$.

5. Discussion and application

5.1. The approximations

The validity of approximations and assumptions made in §2 is considered in appendix C and depends on how the flow is forced (§C.1; e.g. by pressure gradients and thermal radiation). The effects of higher-order terms (§C.2) are negligible only if the eddy coefficients are sufficiently small (C2). A condition that the mean flow can be maintained in the presence of stirring turbulent eddies is that the vertical scale of the latter, determined by the Ozmidov length scale, is much less than that over which the mean flow changes. Included in §C.3 is a discussion of the bounds consequently placed on the vertical diffusivity of the fluid and on the rate of dissipation of turbulent kinetic energy per unit mass, e , using values that apply in the ocean.

5.2. Effects of non-zero viscosity and diffusivity

As shown by (2.8), to the present order of approximation the phase speed of neutral disturbances is not changed by the presence of small viscosity and diffusivity. General formulae for the modified growth rate of small disturbances, kc_{1i} , on the neutral curve of the inviscid and non-diffusive flow are given by (2.9), (3.2) and (3.3) in terms of the eddy coefficients, which may vary with z , and the neutral eigenfunctions, ϕ_1 , solutions of the Taylor–Goldstein equation, (2.5), for periodic two-dimensional disturbances to the mean flow.

We consider first the Holmboe flow. Examples with various forms of the eddy coefficients are given in table 1. Two general patterns are evident. First, the largest growth or decay rates result from the effects of viscosity and diffusion that are non-zero at the mean flow inflection point at $z = 0$, i.e. those that are constant or $\propto \text{sech}^2 az$. Second, for a given value of a coefficient, the effects of vertical viscosity or diffusivity exceed those of the horizontal by about an order of magnitude. (Horizontal eddy coefficients, typically $1 \text{ m}^2 \text{ s}^{-1}$ or more in the ocean, are however much larger than the vertical, typically $O(10^{-2} \text{ m}^2 \text{ s}^{-1})$, compensating this difference.) For the constant and $\text{sech}^2 az$ profiles where the effects on growth are greatest, viscosity acts to damp instabilities as expected, while diffusivity enhances growth. The latter may be because stable density gradients require a growing disturbance to do work against gravity, reducing growth rates and, since diffusion acts to reduce density gradients, it permits relatively greater growth rates.

Vertical eddy coefficients have a variety of effects; the tendency for viscosity (diffusivity) to stabilize (destabilize) the mean flow is not universal. For example a vertical eddy coefficient of viscosity of the form $A_V(z) = A_{V0} \tanh^2 az$ or $A_{V0} \tanh^4 az$ leads to positive growth rates (instability) and, when the coefficient is of the form $A_V(z) = A_{V0} \tanh^2 az \text{sech}^2 az$, whether or not viscosity leads to positive or negative growth rates depends on the wavenumber of the disturbance. Likewise, vertical diffusion may result in growth (e.g. when K_V is constant or $\propto \text{sech}^2 az$) or damping (e.g. when $K_V \propto \tanh^2 az$ or $\tanh^4 az$), respectively destabilizing or stabilizing a flow that is previously neutral. Similarly, various effects on stability are found for horizontal viscosity and diffusivity. Although the effects of viscosity and diffusivity are often opposite, e.g. when the horizontal coefficients are constant, this is not always the case, e.g. when A_H and $K_H \propto \tanh^2 az$: a vertical variation of the eddy coefficients may

change the sign of their effect on disturbances. The additive nature of the effects of the different coefficients in (2.6) and (2.9) implies that the combined effects of vertical and horizontal, viscous and diffusive, coefficients is also additive, (3.2) and (3.3).

Table 2 shows growth rates for the Drazin flow. In two cases (where K_{H0} is constant and for $A_{H0}\tanh^2az \operatorname{sech}^2az$), $\kappa f_{KH}(\kappa)$ or $\kappa f_{AH}(\kappa) = 0$ and the flow on the inviscid and non-diffusive neutral curve remains neutral in the presence of viscous or diffusive effects. This is a change from the Holmboe flow. In the majority of cases, however, the effects of viscosity or diffusivity on stability or instability are similar to those in the Holmboe flow. The exceptions are marked with symbol * in table 2. Notable among these are the reduction or change in growth and decay rates caused by non-zero diffusion for K_{V0} (constant), $K_{V0}\tanh^2az$ and $K_{V0} \operatorname{sech}^2az$; the removal of the Holmboe density interface appears frequently to change the effect of vertical diffusivity on the flow.

No simple rule for the effect of introducing small viscosity or diffusivity – or of weak turbulence – on flows that are neutral in their absence appears to hold. One such rule, our conjecture (§ 1) that viscosity is always stabilizing and that diffusivity is destabilizing, is refuted. Ongoing research is aimed at elucidating the physical mechanisms behind stabilization and destabilization, and thereby attaining the capacity to predict whether a given flow will be stabilized/destabilized by the addition of viscosity/diffusion.

5.3. The critical value of Ri

We have so far examined the variation of growth rate, σ , with viscous or diffusion coefficients at fixed k on the inviscid and non-diffusive neutral curve, (1.2) and (1.4). Small viscosity and diffusivity affects the neutral stability of a stratified shear flow and therefore changes the Richardson number at which disturbances of a given wavenumber become unstable. Howard (1963) devises a useful technique for examining the growth rate of small disturbances to a steady inviscid non-diffusive flow near its neutral curve. If the growth rate on the neutral curve decreases as the minimum Richardson number, J , increases at given k , the neutral curve is an upper boundary in J marking the transition between unstable and stable flow. The Holmboe and Drazin flows are used by Howard as examples. For the Holmboe flow he shows that, near the neutral curve $J = J_c$, the growth rate of disturbances of positive wavenumber, k , is

$$kc_{0i} = -aU_0 \frac{B(\kappa, 1/2)}{\pi} (J - J_c), \quad (5.1)$$

and for the Drazin flow

$$kc_{0i} = -aU_0 \frac{B(\kappa^2, 1/2)}{2\pi\kappa} (J - J_c) \quad (5.2)$$

(after correction by a factor 1/2), where B is the beta function (Abramowitz & Stegun 1965, § 6.2). $B(\kappa, 1/2)$ is positive if, as assumed, $\kappa > 0$, and the growth rate is therefore negative (the flows are stable) if $J > J_c$ and positive (the flows are unstable) if $J < J_c$, as expected. In some cases, but not those considered here, Howard's (1963) technique breaks down (Huppert 1973; but see also Banks, Drazin & Zaturka 1976).

We may therefore extend the study of the growth of disturbances into J space: allowing for variation in J , to order δ the growth rate (defined in § 2) is equal to δkc_{1i} , given by (3.2), plus kc_{0i} , given by (5.1) or (5.2). For example, we consider a Holmboe flow with vertical eddy coefficients $\propto \tanh^2az$. The vertical viscous

coefficient $A_{V0}\tanh^2 az$ has the growth factor in table 1, $\kappa f_{AV}(\kappa) = \kappa$, positive and destabilizing. Vertical diffusivity, $K_{V0}\tanh^2 az$, has a relatively small, but stabilizing, growth factor, $\kappa f_{AV}(\kappa) = -\kappa(1 - \kappa)/2$. The net growth rate of disturbances is therefore

$$k(c_{0i} + \delta c_{1i}) = a^2 A_{V0} \kappa - a^2 K_{V0} \kappa (1 - \kappa)/2 - a U_0 \frac{B(\kappa, 1/2)}{\pi} (J - J_c). \quad (5.3)$$

At $\kappa = 1/2$, $B(\kappa, 1/2)/\pi = 1$, and the growth rate becomes

$$k(c_{0i} + \delta c_{1i}) = 0.5a^2 A_{V0} - 0.125a^2 K_{V0} - a U_0 (J - 1/4). \quad (5.4)$$

This is zero when

$$J = 1/4 + 0.5R_V^{-1} (1 - 0.25Pr_V^{-1}) \quad (5.5)$$

where $Pr_V = A_{V0}/K_{V0}$, is the vertical Prandtl number. The molecular Prandtl numbers in the ocean are ~ 7 in thermally stratified water and 670 for salt stratified water, whilst it is ~ 0.73 for air. The turbulent Prandtl number is of order unity. (According to Kantha & Clayson 2000 the range of the turbulent Prandtl number is ~ 0.7 – 0.9 , but subsequent evidence suggests a value of about $1 + 5J$; Smyth *et al.* 2014.) Generally, therefore, $Pr_V > 0.25$ and the Richardson number, J , of zero growth rate (5.5) exceeds $1/4$; positive growth, or instability, will occur at a minimum Richardson number, J , greater than $1/4$, i.e. when $1/4 \leq J < 1/4 + 0.5R_V^{-1}(1 - 0.25Pr_V^{-1})$. The critical Richardson number is therefore increased above the value of $1/4$ that, from the Miles–Howard theorem (Howard 1961; Miles 1961), is the maximum value of J at which inviscid and non-diffusive flows can become unstable.

This result implies that a stratified laminar shear interface between two weakly turbulent uniform layers having vertical eddy diffusion coefficients that vary like $\tanh^2 az$ is generally unstable in a range of J exceeding $1/4$. Consequently, for example, the interfaces on which billows are observed to form in the Mediterranean thermocline by Woods (1968) may become unstable when the minimum Richardson number, J , is greater than $1/4$, the critical Richardson number of an inviscid, non-diffusive Holmboe flow.

As a counter example, again for the Holmboe flow, the growth factor is $-\kappa(1 + \kappa^2)/2$ when the horizontal viscosity is $A_{H0} \operatorname{sech}^2 az$ (table 1), so the net growth rate when other coefficients are negligible is

$$k(c_{0i} + \delta c_{1i}) = -a^2 A_{H0} \kappa (1 + \kappa^2)/2 - a U_0 \frac{B(\kappa, 1/2)}{\pi} (J - J_c). \quad (5.6)$$

At $\kappa = 0.5$, the growth rate is zero when $J = 1/4 - (5/16)R_H^{-1}$; the critical Richardson number is less than $1/4$. (In this, and the previous case, the parameters $R_V^{-1} = aA_{V0}/U_0$ and $R_H^{-1} = aA_{H0}/U_0$ that appear in the expressions for J must however be small; see § C.2.)

5.4. *Ri* after the collapse of Kelvin–Helmholtz turbulence

Might the viscous and diffusive effects of turbulence explain why the minimum Richardson number, Ri_F , in the laminar shear flow following the decay of turbulence generated by Kelvin–Helmholtz instability exceeds the critical Richardson number, $1/4$? In laboratory experiments, Thorpe (1973) finds a mean $Ri_F = 0.322 \pm 0.063$ while Smyth & Moum's (2000) numerical study gives $Ri_F \sim 1/3$. Is the flow with decaying turbulence still unstable for minimum Richardson numbers, J , up to $\sim 1/3$, but stable at higher J ?

The mean flow during the final stages of turbulent collapse is roughly consistent with the Holmboe flow. If the collapsing turbulence in the centre of the interfacial layer is represented by A_V and $K_V \propto \text{sech}^2 az$, and use is made of (3.3) and values from table 1, the growth rate of a disturbance of non-dimensional wavenumber $\kappa = 0.5$ corresponding to $J_c = 1/4$ at the peak of the inviscid and non-diffusive neutral curve is approximately

$$\sigma = R_V^{-1} (-4.28 + 1.82Pr_V^{-1}). \quad (5.7)$$

(The effects of horizontal viscosity and diffusivity are disregarded.) With a Prandtl number, Pr_V , of order unity, this implies that, because disturbances have negative growth rate, the flow will be stable for Richardson numbers $>1/4$. However, the continued mixing observed at the upper and lower boundaries of the collapsing turbulent region (e.g. see Thorpe 1973; figures 14 and 1e) suggests that A_V and K_V might be better represented in the form $\tanh^2 az \text{sech}^2 az$, having maxima at $z = \pm 0.883a^{-1}$ and decaying to zero at $z = 0$ and as z tends to $\pm\infty$. This choice gives

$$\sigma = R_V^{-1} (0.2188 + 0.1563Pr_V^{-1}). \quad (5.8)$$

The growth rate is now positive and the flow is unstable at $J_c = 1/4$, indicating that zero growth rate will be found at a greater Richardson number or that the minimum Richardson number at which the flow ceases to be unstable exceeds $1/4$.

Although ignoring the time evolution of the mean flow during the collapse of turbulence, does this imply a zero growth rate, i.e. flow stabilization, at $J = Ri_F \sim 1/3$? Following the discussion leading to (5.5) and assuming, as before, that Pr_V is about unity leads to a zero growth rate of disturbances to the flow at $J \sim 1/3$ provided $R_V \sim 4.5$. Although this value of the turbulent vertical Reynolds number, R_V , is somewhat beyond the range of validity of the theory, it offers promise that numerical methods, presently in hand and not subject to the condition (C2), may lead to useful insights into the value of J when turbulence collapses.

Turbulence, represented by eddy coefficients of viscosity and diffusivity, may change the critical Richardson number of a mean flow but whether its effect is stabilizing or destabilizing depends on the vertical variation of the turbulence and on the relative magnitudes of the viscous and diffusion coefficients.

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Appendix A. Singular integrals

Howard (1963) shows that care is needed in evaluating the integrals of the form appearing in (2.9) as a critical layer, $U(z_c) = c_{0r}$, is approached; many of the integrals become singular at this level. Howard finds, however, that in the limit as c_{0i} tends to zero, supposing that $U(z)$ is odd and a monotonically increasing function of z but with $U' \neq 0$ at the critical level and when N^2 is even and $c_{0r} = 0$ (cases exemplified by (1.1)

and (1.3)), terms in (2.9) can be simplified using

$$\lim_{c_{0i} \rightarrow 0} \int_0^\infty U'' W^{-2n} H^2 dz = (1 - \exp(2\pi ni)) \int_0^\infty U'' U^{-2n} H_s^2 dz, \tag{A 1}$$

$$\lim_{c_{0i} \rightarrow 0} \int_0^\infty W^{-(1+2n)} N^2 H^2 dz = [(1 - \exp(2\pi ni))/2n] \int_0^\infty \left(\frac{N^2 H_s^2}{U'} \right)' U^{-2n} dz, \tag{A 2}$$

where $H = \phi_1 W^{n-1}$ ($=H_s$ at $z = z_c$) and $n = \kappa$ for the Holmboe flow and κ^2 for the Drazin flow. For example, (A 2) may be used to express the indefinite integral of the form $\lim_{-\infty}^\infty \sinh^{-(1+2\kappa)} z \operatorname{sech} z dz$, that appears in I when the Holmboe form of the stream function is inserted, leading to

$$I = -k^2 \pi [1 - \exp(2\pi \kappa i)] / (\sin \kappa \pi). \tag{A 3}$$

The integral,

$$\int_0^\infty \sinh^{-(1+2\kappa)} z \operatorname{sech} z dz, \tag{A 4}$$

and other integrals in (2.9) are evaluated using beta and gamma functions (e.g. see Abramowitz & Stegun 1965: especially equations 6.1.15, 6.1.17, 6.2.1) and 6.2.2). For the Drazin flow,

$$I = -k^2 \pi [1 - \exp(2\pi \kappa^2 i)] / (\sin \kappa^2 \pi). \tag{A 5}$$

Appendix B. Numerical methods

The numerical tests of the perturbation theory require computation of the growth rate, σ , at $J_c = 0.25$, and at wavenumbers $\kappa = 0.5$ or 0.707 for the Holmboe and Drazin flows, respectively, for various profiles of the viscosity and diffusivity, and examining the behaviour as the amplitude of the viscosity and diffusivities, represented by C_{eff} , say, tend to zero.

The numerical calculation of σ employs the matrix method of Liu *et al.* (2012) to find normal mode solutions of (2.1) and (2.2). Impermeable free-slip constant-density boundary conditions are imposed at $az = \pm 5$ to approximate the infinite depth assumed in the theory. The matrix method is accurate to second-order in the non-dimensional grid spacing, set here to 0.01.

Profiles of the eddy coefficients are chosen as in tables 1 and 2. In vertical cases where viscosity vanishes at $z = 0$, the viscosity profile is supplemented with a small constant value, ΔA_V (equal to 10^{-4} in non-dimensional form), to improve numerical stability. This adjustment has only a minor effect on the results.

The matrix method fails close to stability boundary and we therefore perform the analysis at $J = 0.20, 0.21, 0.22$ and 0.23 , $\kappa = 0.5$, and extrapolate to $J = 0.25$ by fitting a parabola to the resulting growth rates. This procedure is repeated at several small values of the eddy coefficient, C_{eff} , the extrapolated growth rates being fitted, again by a parabola, to determine $\partial\sigma/\partial C_{\text{eff}}$, equivalent to the quantities $\kappa f_{AV}(\kappa)$ etc. in (3.2) and (3.3).

Sensitivity of the estimated growth rates to the non-dimensional vertical separation, H (in units of a^{-1}), of the boundaries is relatively small. For example, $\partial[\kappa f_{AV}(\kappa)]/\partial H \sim -0.019$ at $\kappa f_{AV}(\kappa) = 0.5520$, the value given in table 1 when $H = 10$, $\kappa = 0.5$ and $A_V = A_{V0} \tanh^2 az$, and $\partial[\kappa f_{KV}(\kappa)]/\partial H \sim 0.0048$ at $\kappa f_{KV}(\kappa) = -0.1214$ as in table 1 when $H = 10$, $\kappa = 0.5$ and $K_V = K_{V0} \tanh^2 az$.

Appendix C. The validity of approximations

C.1. Changes to the mean flow and density field

The validity of the approximations made in §2 can be assessed in view of the magnitude of the growth rates found in §3 for the Holmboe and Drazin flows. In §2 it is assumed that the rate of change of the mean flow and density field are much smaller than that of a small disturbance. Only vertical viscosity or diffusivity may change the mean flow and need be considered. Smyth *et al.* (2011) show that three conditions are necessary for the mean flow to be regarded as steady: the pressure gradient must balance the action of viscosity; the mean vertical pressure gradient must be in hydrostatic balance with the mean density; and the action of diffusion must not change the mean density profile significantly on the time scale of the change of a small disturbance. The first two of these conditions may be assumed to hold in physically realizable situations. The third condition holds automatically for the Drazin flow when diffusivity is uniform but is less likely to be valid for the Holmboe flow or when diffusivity varies in z . The mean density, $\Pi(z, t)$, is given by a conservation equation of the form

$$\frac{\partial \Pi}{\partial t} = \frac{\partial}{\partial z} \left(K_V \frac{\partial \Pi}{\partial z} \right). \quad (\text{C } 1)$$

If the vertical scale of the density variation and of the vertical diffusivity, K_V , is a^{-1} , and the vertical diffusivity is characterized by K_{V0} (as in the examples taken in §3), the mean density, and therefore the buoyancy frequency, will change at a rate $a^2 K_{V0}$. This is about the same rate as that of disturbances to the diffusive flow (see (3.2)) and the disregarding of the change to the mean flow is invalid.

We might escape the problem either by introducing a buoyancy flux that balances the diffusion term on the right of (C 1) (e.g. through a radiation term that modifies the mean fluid density, much as did Matthews 1988 in regard to density in a frozen lake, but in general this is a rather contrived situation) or by restricting attention to cases in which vertical diffusion results in changes to the mean flow at a rate much less than the viscosity-induced changes to disturbances. Results for the Holmboe flow are now being tested using numerical methods that are valid for larger growth rates. Preliminary results tend to substantiate the qualitative predictions (stabilization of destabilization) in table 1.

C.2. Higher-order terms

The validity of comparing terms with coefficients of order ε or of $\varepsilon\delta$, and disregarding those of order ε^2 , can be assessed by comparing the magnitude of terms in (2.1) and (2.2). The $\varepsilon\delta$ terms involving the viscous and diffusion coefficients, of order C_{oeff} , say (representing A_{V0} , K_{H0} etc. as in appendix B), will be small compared to the ε terms provided

$$\chi = \frac{C_{\text{oeff}}}{U_0} \ll 1 \quad (\text{C } 2)$$

(χ is equal to the inverse Reynolds number when $C_{\text{oeff}} = A_{V0}$). Equation (C 2) can be evaluated noting that $2a^{-1}$ is a measure of the interface thickness: the scale $a \sim 10^2 \text{ m}^{-1}$ in laboratory experiments corresponds to an interface thickness of $2 \times 10^{-2} \text{ m}$. In a laminar laboratory flow with $Pr > 1$ (e.g. heat or salt stratification of water) the kinematic viscosity, $\nu \sim 10^{-6} \text{ m}^2 \text{ s}^{-1}$, exceeds the diffusion coefficient. Taking $U_0 \sim 10^{-1} \text{ m s}^{-1}$ gives $\chi \sim 10^{-3} \ll 1$. In the ocean representative scales

might be an interface thickness of 20 m (so $a \sim 10^{-1} \text{ m}^{-1}$) and $U_0 \sim 10^{-1} \text{ m s}^{-1}$, so that $\chi \ll 1$ if $C_{\text{oeff}} \ll 1 \text{ m}^2 \text{ s}^{-1}$. This is likely to be valid for the vertical coefficients, A_V and K_V , but may be less likely for horizontal turbulent eddy coefficients, A_H and K_H . The narrow sheet-like interfaces observed by Woods (1968) are of thickness $\sim 0.2 \text{ m}$ ($a \sim 10 \text{ m}^{-1}$) and $U_0 \sim 1 \times 10^{-2} \text{ m s}^{-1}$, so that $\chi \ll 1$ if $C_{\text{oeff}} \ll 1 \times 10^{-3} \text{ m}^2 \text{ s}^{-1}$.

If η is the perturbation of density surfaces caused by a displacement with

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \eta = -\frac{\partial \psi}{\partial x}, \quad (\text{C } 3)$$

the magnitude of the vertical displacements, A_{mp} , resulting from a stream function of amplitude A is of order A/U_0 . The (Im) terms of order ε^2 in (2.1) and (2.2) will be small compared to the $\varepsilon\delta$ terms if $C_{\text{oeff}}/(U_0 A_{mp}) \gg 1$. Noting (C 2), it follows that $A_{mp} \ll a^{-1}$, or that the neglect of ε^2 terms is valid only whilst disturbances are of amplitude much less than the interface thickness.

C.3. Effect of turbulence in the mean flow

A condition for the consistency of selecting a steady mean flow, $U(z) = U_0 \tanh az$, in the presence of turbulence with an overturning scale equal to the Ozmidov scale, L_O , is that the mean flow velocity length scale, $[(dU/dz)/U_0]^{-1} \gg L_O$. (A similar condition applies for the mean density.) Now $L_O = e^{1/2}/N^{3/2}$, where e is the rate of dissipation of turbulent kinetic energy per unit mass, and hence the condition becomes

$$e \ll \frac{N^3}{a^2} \cosh^4 az. \quad (\text{C } 4)$$

For both the Holmboe and Drazin flows (C 4) is satisfied for all z if $e \ll N_0^3 a^{-2}$.

The Osborn (1980) formula, $K_V = \gamma e/N^2$, with $\gamma \approx 0.2$ then leads to the condition

$$K_V \ll 0.2 \frac{N_0}{a^2}. \quad (\text{C } 5)$$

For an oceanic interface with a fractional buoyancy difference ($g\Delta\rho/\rho_0$) of 10^{-4} m s^{-2} corresponding approximately to a temperature difference of 0.1 K at a temperature of $\sim 5 \text{ }^\circ\text{C}$, $N_0 = (ag\Delta\rho/\rho_0)^{1/2} \approx 3.2 \times 10^{-3} \text{ s}^{-1}$ if $a \sim 10^{-1} \text{ m}^{-1}$ (a 20 m thick interface), so that e and K_V must be much less than $3.3 \times 10^{-6} \text{ m}^2 \text{ s}^{-3}$ (greater than the largest values commonly observed, e.g. Liu *et al.* 2012), and $6.4 \times 10^{-2} \text{ m}^2 \text{ s}^{-1}$, respectively. For the thin thermally stratified ‘sheet’ interfaces observed by Woods (1968) the corresponding values are $3.3 \times 10^{-7} \text{ m}^2 \text{ s}^{-3}$ and $6.4 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}$, respectively. The latter bound on K_V may possibly be satisfied in view of Wood’s observation that flow in the sheets appears to be almost laminar and therefore with a diffusion coefficient approaching the molecular value for heat of about $1.4 \times 10^{-7} \text{ m}^2 \text{ s}^{-1}$.

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