SHEAR-DRIVEN OVERTURNS

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Shear-driven overturns are roughly linear concentrations of vorticity that lead to turbulence and mixing in sheared flows. Banded clouds are a well-known atmospheric manifestation of this phenomenon. Shear-driven overturns are important in virtually all geophysical flows, and in many engineering applications as well.

The term “overturn” usually indicates the overturning of isopycnals by shear instability in a stably stratified fluid. In that case, turbulent mixing drives a downgradient mass transport, and thereby raises the gravitational potential energy of the fluid. In this article, I will describe the process as it occurs in both homogeneous and stably stratified flows.

1. THE PRIMARY INSTABILITY

Overturns begin as a dynamic instability fed by the kinetic energy of a parallel shear layer. This mechanism is commonly called Kelvin-Helmholtz instability, though that designation is historically correct only in the stratified case.

1a) THE MECHANISM

Imagine two fluid layers moving opposite to one another, separated by a thin transition layer (figure 1a). For the moment, assume that density is uniform so that buoyancy effects are negligible. The vorticity is concentrated within the transition layer. Now suppose that the transition layer suffers a slight, sinusoidal displacement in the vertical. This creates constrictions in which flow must accelerate. The accelerated flow preferentially advects the vorticity inward toward the convergence point (located at the center of the frame). The resulting concentration of vorticity (figure 1b) induces enhanced clockwise motion around the convergence point, including vertical motions that amplify the original sinusoidal displacement. The result is a positive feedback loop that leads to exponentially-growing instability.

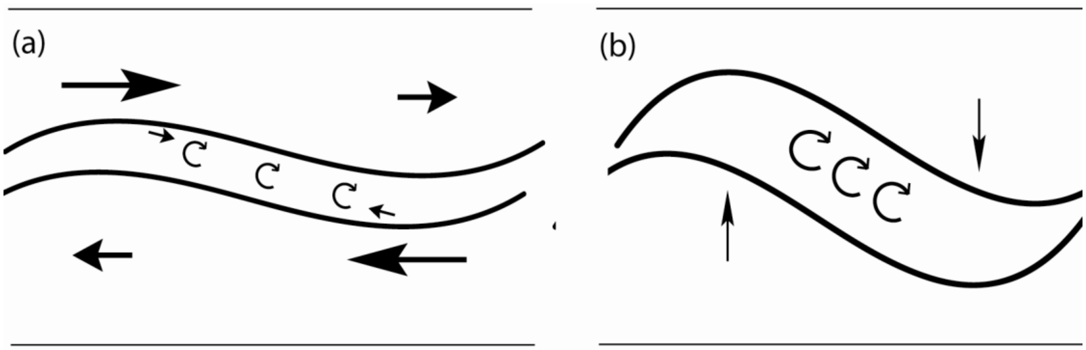


Figure 1: Schematic representation of the early stages in the rollup of a shear layer.

1b) PARAMETER DEFINITIONS

In the simplest model, a train of shear-driven overturns begins as a stably stratified, parallel shear flow as shown in figure 2. These profiles represent similarity solutions of the one dimensional diffusion equation, and have the analytical representation[1]

$$\frac{U}{\Delta u} = -\frac{\rho - \rho_0}{\Delta \rho} = \frac{1}{2}\mathrm{erf}\left(\frac{\sqrt{\pi}}{2}\frac{z}{h(t)}\right). \tag{1}$$

As the overturns grow, (1) may be interpreted as an approximate model for the horizontally averaged profiles. The thickness h of the transition layer increases in time from its initial value h_0. The property differentials Δu and $\Delta\rho$ are constant.

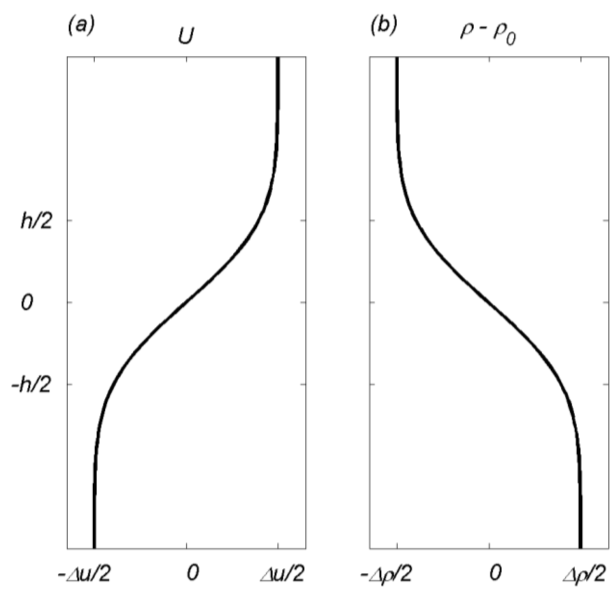


Figure 2: Profiles describing a stably stratified parallel shear layer. (a) Horizontal velocity. (b) Density.

[1] The precise form of the profiles is not important.

The subsequent evolution is determined by three nondimensional parameters. The competition between the destabilizing influence of shear and the stabilizing effect of buoyancy is expressed in terms of the gradient Richardson number:

$$Ri(z) = \frac{-g}{\rho_0}\frac{d\rho/dz}{(du/dz)^2}$$

where g is the gravitational acceleration. It is often convenient to employ a single "bulk" value for the Richardson number. In this case, the simplest choice is the minimum value that occurs at the center of the shear layer:

$$Ri_{\min} = \frac{g\Delta\rho}{\rho_0}\frac{h}{\Delta u^2}.$$

In the inviscid limit, a necessary conditions for instability is $Ri_{\min} < 1/4$ (Miles 1961, Howard 1961). Molecular effects are described by the macroscale Reynolds number

$$Re = \frac{h\Delta u}{\nu}$$

and the Prandtl number

$$Pr = \frac{\nu}{\kappa}.$$

where ν and κ represent the kinematic viscosity and the mass diffusivity, respectively.[2] Note that both the minimum Richardson number and the Reynolds number increase in proportion to the thickness of the transition layer, while the Prandtl number remains constant.

1c) LENGTH AND TIME SCALES

Unstable modes are stationary in the reference frame moving with the center of the transition layer. The preferred wavelength for the instability is approximately as shown on figure 1, about $7h_0$. In homogeneous flow, the exponential growth rate σ is approximately $0.2\Delta u/h_0$. To visualize this growth rate, imagine a particle traveling in either the upper or the lower stream. In the time it takes the particle to traverse one wavelength of the instability, the amplitude of the latter grows by about a factor of twenty. In most natural flows, the shear is not steady. A useful criterion for the development of instability is that the shear persist over a time much longer than the e-folding time, σ^{-1}.

[2] We assume that density is controlled by a single scalar constituent, e.g. temperature, so that the mass diffusivity is well defined.

1d) BUOYANCY EFFECTS

In stable stratification, overturning requires that work be done against gravity, which therefore limits the growth process described above. If the minimum Richardson number $Ri_{\min}$ is positive but less than ¼, the flow is unstable but with growth rate reduced below the value found in homogeneous flow. A reasonable approximation is

$$\sigma = \begin{cases} 0.2\dfrac{\Delta u}{h_0}\left(1-4Ri_{\min}\right) & \text{if} \quad 0 \le Ri_{\min} < 1/4 \\ 0 & \text{if} \quad Ri_{\min} \ge 1/4 \end{cases}$$

The wavelength of the most amplified mode remains near $7h_0$ regardless of stratification. Both of these approximations are accurate to within about 10%. In a slowly-varying mean flow, the condition $Ri_{\min} < 1/4$ must persist over a time much longer than σ^{-1} for instability to grow appreciably.

1e) BOUNDARY EFFECTS

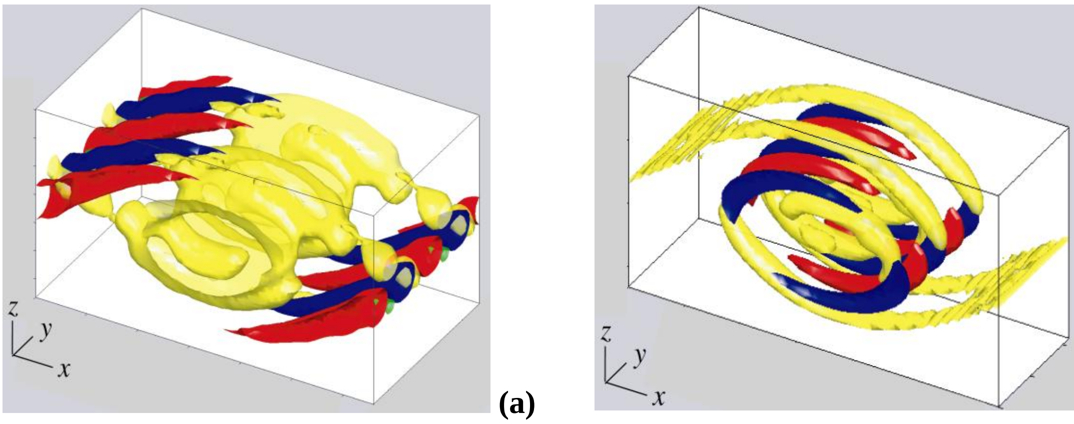
If upper and lower boundaries are located at least one wavelength of the FGM ($\sim 7h_0$) above and below the transition layer, they will have minimal effect on the primary overturning instability. Boundaries placed closer than this reduce the growth rates and displace the growth rate maximum slightly toward longer waves. Asymmetrically placed boundaries can radically alter the nature of the instability.

1f) VISCOSITY & DIFFUSION

Molecular effects have negligible influence on the primary instability provided that the Reynolds number exceeds $\sim O(10^2)$. At lower Reynolds numbers, molecular effects on the mean flow, as well as those affecting the instability, must be considered.

2. SECONDARY INSTABILITY AND TRANSITION

The emergence of three-dimensional secondary circulations is due to one of several secondary instabilities. These typically feature streamwise vortices (e.g. figure 3). In homogeneous flow (figure 3a), secondary instability is focused in the braid regions between the billows and is driven largely by vortex stretching (Lasheras et. al. 1986). Note also the appearance of “cups” above and below the billow cores (Rogers & Moser 1994).



(a) **(b)**

Figure 3: Eigenfunctions of the vorticity, $\vec{\omega}$, for the fastest-growing secondary instabilities of a mature KH billow. Red and blue indicate isosurfaces of the vorticity in the *x*-*z* plane, i.e. $(\omega_x^2 + \omega_z^2)^{1/2}$, colored using the sign of ω_x. Yellow is the total (background plus perturbation) spanwise vorticity ω_y, indicating the position of the primary billow. **(a)** Strain-driven mode focused in the braid regions. **(b)** Convective instability focused in the overturned regions of the billow. The counterrotating line vortices are delineated by the red and blue isosurfaces. Eigenmodes are computed as in Klaassen & Peltier (1991).

In the stratified case, the vortex stretching mechanism is reinforced by convective instability as the stable density stratification is overturned (Klaassen & Peltier 1991). This leads to a preference for instability in the overturned regions of the billows (Figure 3b, 4a).

The braid regions are susceptible to an additional instability that is called secondary KH instability because of its visual resemblance to the primary KH instability (figure 4b). This mode becomes more vigorous as the Reynolds number increases. After the emergence of these secondary instabilities, the flow evolves first into a train of turbulent billows (figure 4c) and finally into a layer of homogeneous turbulence (figure 4d), which decays gradually under the action of viscosity and thermal diffusion.

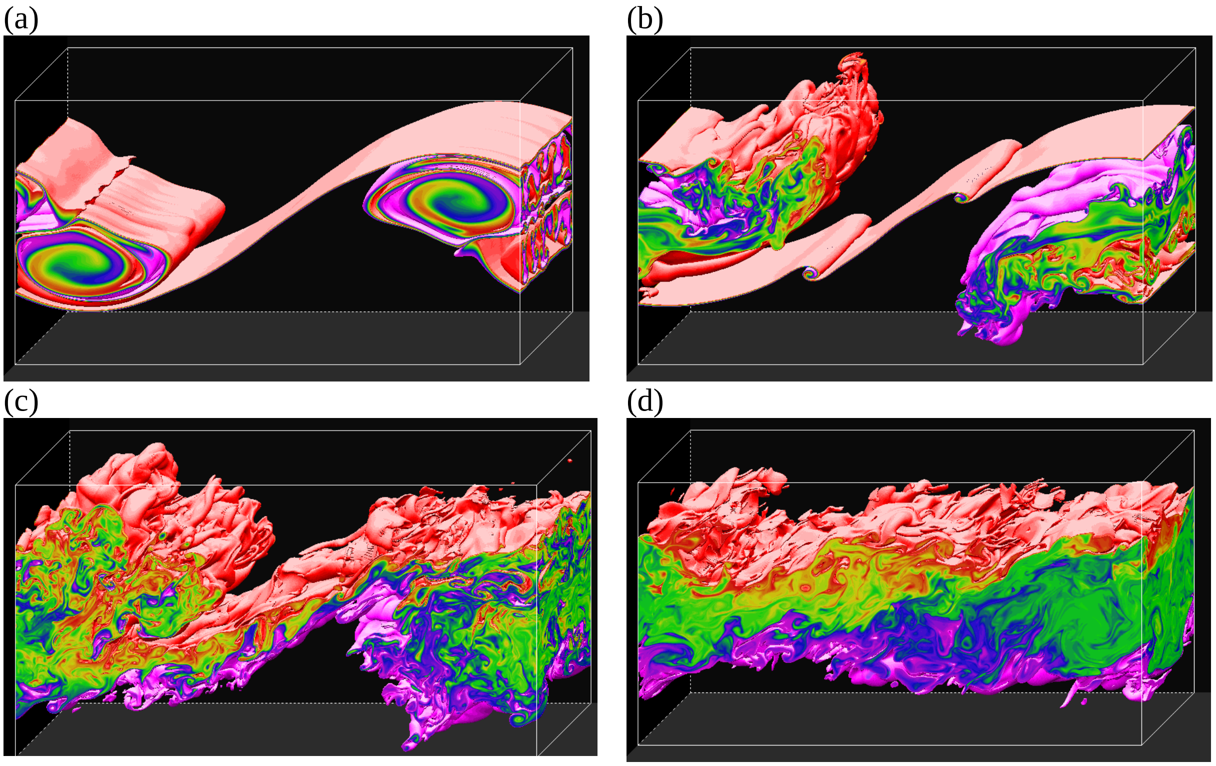


Figure 4: Snapshots of the temperature field from a direct numerical simulation of shear-driven overturns in thermally stratified water. The initial minimum Richardson number was $Ri_0 = 0.08$, and the Reynolds number based on the full thickness and velocity change across the shear layer was $h_0 \Delta u / \nu = 800$. Times, given in units of $h_0 / \Delta u$, are **(a)** t=43, **(b)** t=62, **(c)** t=84, **(d)** t=134.

3. TURBULENCE

3a) SELF-SIMILAR GROWTH IN HOMOGENEOUS FLOW

The instability described above causes a shear layer to roll up into a train of discrete vortices. Subsequently, these vortices amalgamate to form larger vortices. In the absence of buoyancy effects, this merging process continues until it is ultimately suppressed by boundaries.

At this point we must make a distinction between temporally growing and spatially growing instabilities. The former is statistically homogeneous (or periodic) in the streamwise direction and evolves in time. In the spatially growing case, the flow is statistically stationary. The instability originates at a specific location in space and grows in the downstream direction. Temporal growth is usually a good model for overturns occurring far from boundaries, and is somewhat easier to manage analytically. Spatial growth occurs near boundaries, the canonical example being the splitter plate used in lab experiments (e.g. Koop & Browand 1979). As a first approximation, the two can be related by a simple Galilean transformation; however at more advanced stages (in both evolution and analysis) the difference must be considered more carefully. Here, we will focus specifically on temporal growth.

During the merging process, an average taken in the homogeneous coordinate shows that the "transition layer" thickens more-or-less monotonically as fluid from the upper and lower layers is entrained. The process is approximately self-similar in the sense that each doubling of the average vortex size (and thus of the transition layer thickness) takes about twice as long as the previous doubling, so that the layer thickness grows linearly in time.

Besides merging, secondary instabilities as described above may act to produce three-dimensional circulations and thus to drive the flow toward the turbulent state. The complications of three-dimensionality and turbulence do not prevent the layer thickness from growing linearly. This linear growth is represented by an entrainment velocity proportional to the velocity differential across the shear layer, i.e. $w_e \equiv dh/dt = S\Delta u$. The spreading rate is given by $S = 0.065 \pm 0.005$[3].

3b) TURBULENCE DECAY IN THE STRATIFIED CASE

In the stratified case, thickening of the transition layer causes the Richardson number to increase with time. Eventually, Ri_{min} exceeds ¼ and the instability that drives turbulence no longer exists. Turbulence begins to decay near this time. The end result is a new parallel shear flow that is stable in consequence of its increased Richardson number. Both laboratory experiments (Thorpe 1973) and numerical simulations (Smyth & Moum 2000) have shown that end states of turbulent, stratified shear layers are characterized by $Ri_{min} \approx 0.32$.

3c) SOME CHARACTERISTICS OF THE TURBULENT STATE

Stratified turbulence is characterized in terms of classical turbulence theory using the buoyancy Reynolds number: $Re_b = \varepsilon / \nu N^2$, where ε is the rate of kinetic energy dissipation by the turbulence. This quantity represents the squared ratio of the Kolmogorov eddy turnover rate to the buoyancy frequency. Large values indicate that small-scale turbulent motions are independent of the stratification. Turbulence resulting from the breakdown of shear-driven overturns is statistically homogeneous in the horizontal directions. Eddies in the dissipation subrange are isotropic provided that Re_b is greater than $O(10^2)$. Of the kinetic energy released from the mean shear flow during

[3] This estimate is the average of a lab experiment (0.069; Bell & Mehta 1990) and a direct numerical simulation (0.062; Rogers & Moser 1994). This estimate applies to the case of temporally-growing, horizontally periodic overturns; for overturns growing from a fixed point in space, the constant of proportionality is about 50% higher (Champagne et al. 1976).

this turbulent phase, about 20% is expended raising the gravitational potential energy of the fluid[4], while the remaining ~80% is dissipated as heat.

4. FLUXES AND THE TURBULENT DIFFUSIVITY

4a) THE HOMOGENEOUS CASE

The entrainment velocity can be translated into a turbulent diffusivity, from which fluxes of all relevant quantities may be estimated, i.e. $F_X = -K\Delta X / h$ where F_X is the flux of some volumetric concentration X across the transition layer and ΔX is the corresponding differential. Suppose that the mean velocity profile $U(z,t)$ obeys a diffusion equation:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial z}\left(K \frac{\partial U}{\partial z}\right)$$

where K is the turbulent diffusivity. Now assume that the velocity profile expands while retaining its shape, i.e. that $U = U(s)$ where $s = z / h(t)$, and that the difference between the velocities above and below the transition layer retains the constant value Δu. As long as K is independent of z, the vertical dependence of the solution is given by (1).

Suppose further that K has the mixing length form

$$K = a_0 h \Delta u \qquad (2)$$

where a_0 is a constant. This diffusivity is uniform in space but varies in time as the transition layer thickens. One may then show that the entrainment velocity is constant and is given by

$$\frac{dh}{dt} = 2\pi a_0 \Delta u \,.$$

The spreading rates quoted above therefore imply that $a_0 \approx .065 / 2\pi \approx 0.010$. Using these results, the flux of a volumetric concentration X across a homogeneous layer where the velocity change is Δu can be estimated as

$$F_X = -0.010 \Delta u \cdot \Delta X \,.$$

For example, if the two layers differ in temperature by ΔT (where ΔT is small enough that buoyancy effects are unimportant), then the heat flux across the layer is approximately $-0.010 \rho c_p \cdot \Delta u \cdot \Delta T$. Given a temperature difference of *0.01K*, a velocity difference of *0.1m/s* and $\rho c_p = 4 \times 10^6 \, Jm^3 K^{-1}$, we would expect a heat flux of $40 Wm^{-2}$.

[4] This fraction is uncertain by about a factor of two.

4b) THE STRATIFIED CASE

Parameterizing fluxes across a stably stratified, turbulent shear layer is very much an area of active research. A simple formulation uses the mixing length form (2), but adds a Richardson number dependence to represent stratification effects, e.g.

$$K = ah\Delta u\,, \tag{3}$$

where

$$a = a_0 \times \begin{cases} 1, & \text{if } Ri < 0 \\ (1 - Ri_{\min}/Ri_c), & \text{if } 0 < Ri < Ri_c \\ 0, & \text{if } Ri > Ri_c \end{cases} \tag{4}$$

The constant a_0 is as determined above for the homogeneous case $Ri_{\min} = 0$. The critical Richardson number Ri_c is to be determined empirically. The entrainment velocity is no longer constant, but the evolution of h conforms well to observations. Fitting to the DNS run shown in figure 4 yields $Ri_c = 0.32$ and $a_0 = 0.008$. This value of a_0 is within 20% of that found in the homogeneous limit, indicating the Richardson number dependence described in (4) is reasonably accurate. Also, the value 0.32 for Ri_c agrees very closely with lab experiments (Thorpe 1973). Nonetheless, in view of the preliminary nature of this representation of the effects of stratification on turbulent fluxes, one should probably retain the value of a_0 quoted previously. In summary, fluxes across a stratified shear layer are best approximated using (3) and (4), with $a_0 = 0.010$.

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