A.5 OC680 Homework #5 Due Wednesday Feb. 21

1. **The fourth derivative matrix**

   (a) *Derive a 2nd-order finite difference approximation to the fourth derivative having the following form:

   $$f_i^{(4)} = Cf_{i-2} + Bf_{i-1} + Af_i + Bf_{i+1} + Cf_{i+2} \tag{A.5.1}$$

   Hint: To simplify the algebra, write (A.5.1) in this form:

   $$f_i^{(4)} = Af_i + B(f_{i-1} + f_{i+1}) + C(f_{i-2} + f_{i+2}) \tag{A.5.2}$$

   In the Taylor series expansions for the pairs of terms in parenthesis, every second term will cancel.

   (b) In (A.5.1), the expressions for f_1, f_2, f_{N-1} and f_N involve "ghost points" (at which f is not specified). *Explain how these expressions can be evaluated using each of the following boundary conditions (in finite difference form):

   - Rigid boundaries: $f_0 = f_{N+1} = 0\,;\quad f_0' = f_{N+1}' = 0.$
   - Frictionless boundaries: $f_0 = f_{N+1} = 0\,;\quad f_0'' = f_{N+1}'' = 0\,.$

   Note: It is sufficient to express the boundary conditions to second-order accuracy, e.g.

   $$f_0{}' = \frac{f_1 - f_{-1}}{2\Delta} = 0.$$

2. **Matrix solution of the Orr-Sommerfeld equation**

   *Write a matlab function to find eigenvalues and eigenvectors for the discretized Orr-Sommerfeld equation. Your function should accept as inputs a column vector of z values, the corresponding background velocity vector $U(z)$, the viscosity ν, the wavenumbers k, ℓ and a choice of rigid or frictionless boundary conditions at each boundary. It should deliver as output the growth rate and vertical velocity eigenfunction for the fastest-growing mode. The function should compute U'' internally.

   You will need to write a subroutine **ddz4(z)** to compute the 4th derivative using your results from PROJECT #2. That routine need not include 1-sided derivatives at the boundaries (because you won't actually use it to compute the fourth derivative of anything).

   *Try your code for the following test case:

   $$0 < z* < 1\,;\quad \Delta z* = 0.005\,;\quad U* = 4z*(1-z*)\,;\quad \nu* = 1/1e5;\quad k* = 1.55\,;\quad l* = 0.$$

   Refer to your notes on scaling to make sure you understand what the starred variables mean, how they are input to your subroutine, and how to interpret the output. Use rigid boundary conditions. I get $\sigma* = 0.015 - 0.243\imath$. *Suppose you needed to apply this result to a particular channel flow, with width 15m and maximum flow speed 2m/s. Give the wavelength in meters and the e-folding time in seconds (or minutes if that seems more sensible).

3. **Wave resonance in a jet**

   The triangular jet profile shown in figure A.4 has three kinks where vortical waves can propagate. *Sketch the three waves such that each adjacent pair satisfies the criteria for resonance:

   (a) The vertical velocity perturbations of each wave amplify the crests and troughs of the other.

   (b) The propagation velocities allow for the waves to be stationary relative to each other.

   Make your own sketch if you prefer.

   Comparing with your analysis of the Bickley jet in assignment 4, does your sketch represent the sinuous or the varicose mode?

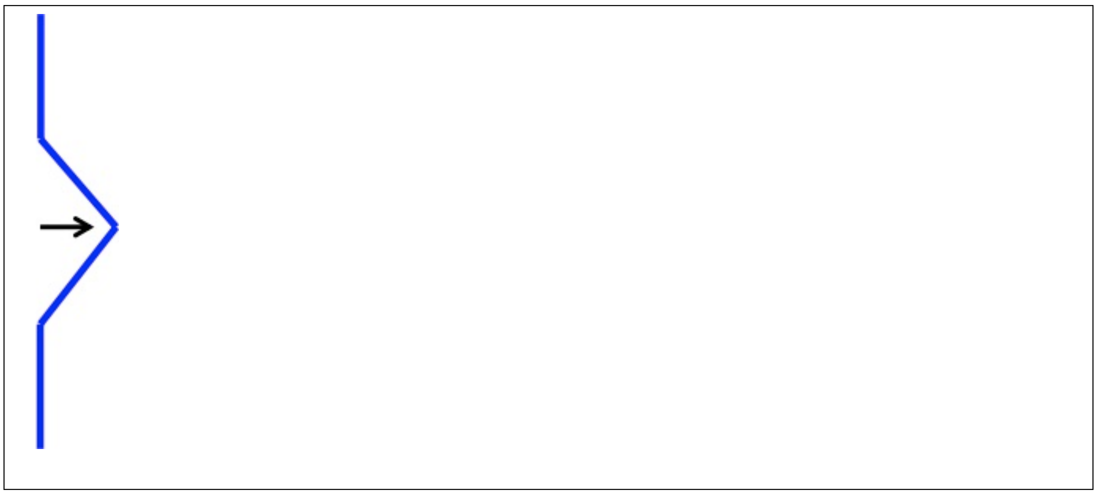

Figure A.4: Schematic velocity profile for a triangular jet.

4. **A convectively unstable layer in an inviscid fluid, revisited**

   In an earlier problem you developed a code to solve the Rayleigh equation. Adapt this code to solve (2.3.10), the equation for convection in a stationary, inviscid fluid with an arbitrary buoyancy profile. You will now use this code to address the unstable layer problem 3 from homework #2 more thoroughly.

   (a) Using the B_z profile (A.2.2), reproduced below for convenience, compute and plot the growth rate for a full range of $\tilde{k}$. For simplicity choose $\alpha = 1$.

$$B_z = B_{z0}(1 - 2\mathrm{sech}^2 \alpha z). \tag{A.5.3}$$

   (b) For the special case $\tilde{k} = \alpha$, do your numerical results for σ and $\hat{w}$ match the analytical solution? Make a plot to illustrate the comparison.

   (c) Does this case represent the fastest-growing mode?

   (d) If $\tilde{k} = \alpha$ is not the fastest-growing mode, compute $\tilde{k}$, σ and $\hat{w}$ for the fastest-growing mode and discuss any differences you observe. Make a plot to illustrate the comparison.