OCE680 Assignment 2, Instructor's notes

1: Benard convection [10]

(a) Differentiating the quadratic with respect to $\cos^2\theta$ gives

$$\frac{\partial\sigma}{\partial\cos^2\theta} = \frac{-B_z}{2\sigma+(\nu+\kappa)K^2}.$$

This is positive assuming that (1) $B_z < 0$, as is normally required for convective instability, and (2) σ is real and positive, which is true for convective instability.

Growth is maximized when $\cos^2\theta$ is at its maximum value, 1.

(b) Define $f = (\tilde{k}^{*2}+n^2\pi^2)^3/\tilde{k}^{*2}$ and differentiate with respect to $\tilde{k}^{*2}$:

$$\frac{\partial f}{\partial\tilde{k}^{*2}} = \frac{(\tilde{k}^{*2}+n^2\pi^2)^2}{\tilde{k}^{*4}}\left[2\tilde{k}^{*2}-n^2\pi^2\right],$$

which is zero only if $\tilde{k}^{*2} = n^2\pi^2/2$, or

$$\tilde{k} = n\pi/\sqrt{2}.$$

In this case

$$f = \frac{27}{4}n^4\pi^4 \approx 657.5 \text{ for } n=1.$$

2: A convective mixed layer [10]

The Rayleigh number is 4.5×10^{12}. The easiest way to compute the fastest-growing mode is by plugging the given parameter values into the quadratic (2.5.10) and solving numerically. Compute σ over a range of wavenumbers, plot and check that your range includes the maximum growth rate, then identify the corresponding wavenumber.

I get a horizontal wavelength of about 5m. The growth rate is $4.5\times10^{-4}s^{-1}$, and the e-folding time, σ^{-1}, is about 34 minutes. The point here is that the e-folding time is short compared with the time over which conditions for instability persist, so that the instability has enough time to grow. If the e-folding time was longer than a few hours, the sun would be up before convection really had a chance to get started.

Here is the code I used:

```
clear
close all

% plotting parameters
fs=20;
lw=1.5;
set(0,'DefaultAxesFontSize',fs)

% problem parameters
nu=1e-6;        % viscosity
kappa=1.4e-7;   % diffusivity
H=40;           % layer thickness
drho=1e-6;      % density difference
g=9.81;         % gravity

% computed parameters
Bz=-g*drho/H;           % buoyancy gradient
Pr=nu/kappa;            % Prandtl number
```

```
Ra=-Bz*H^4/(nu*kappa)   % Rayleigh number

kt_sts=10.^[.5 :.01 :2.8];  %wavenumber values

for n=1:3

    for i=1:length(kt_sts)
        kt_st=kt_sts(i); % choose one value of kt_st
        sigma_st=RB_convection(kt_st,n,1,-Ra*Pr,Pr,1);
        sr_st=max(real(sigma_st));       % pick the larger root
        sr(n,i)=sr_st*kappa/H^2;         % redimensionalize growth rate
    end

    i0=find(sr(n,:)==max(sr(n,:))); % find index for FGM
    kt0=kt_sts(i0)/H;               % redimensionalize wavenumber of FGM
    efold_min(n)=1/sr(n,i0)/60;    % e-folding time of FGM in minutes
    lambda(n)=2*pi/kt0;             % wavelength of FGM in m

end

% plot
figure
semilogx(kt_sts/H,sr,'linewidth',lw);
xlim([.1 20])
h=legend('n=1','n=2','n=3','location','south');
xlabel('kt [m^{-1}]');
ylabel('\sigma [s^{-1}]');
title(sprintf('n=1: e-fold=%.0fmin., \\lambda=%.2fm.',...
    efold_min(1),lambda(1)),'fontweight','normal');
set(gca,'fontsize',fs-2)
```

and subroutine

```
function sigma=RB_convection(kt,n,H,Bz,nu,kappa)
%
% Calculate growth rate for Rayleigh-Benard convection.
% Inputs:
% kt = horizontal projection of wave vector
% n = vertical quantization number
% H = layer thickness
% Bz = buoyancy gradient
% nu = viscosity
% kappa = diffusivity
%
% Outputs
% sigma = growth rate (two values)
%

K2=kt^2+(n*pi/H)^2; % total wave vector length (squared)
% coefficients for quadratic equation
B=(nu+kappa)*K2;
C=nu*kappa*K2.^2+Bz*kt^2./K2;
D=B.^2-4*C; % discriminant
```

```
sigma(1)=.5*( -B+sqrt(D));
sigma(2)=.5*( -B-sqrt(D));
return
```

Results are shown in figure 14.9.

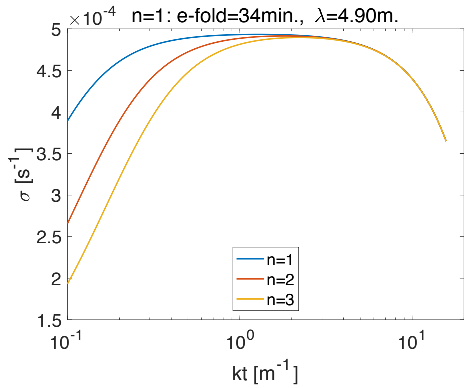

Figure 14.9: Growth rate versus horizontal wavenumber for convective modes $n = 1,2,3$.

3: An unstable layer in an inviscid fluid [10]

Knowing that $\text{sech}^2(0) = 1$ and $\text{sech}^2(\pm\infty) = 0$, we can tell that B_z goes to B_{z0} far from $z = 0$ and $-B_{z0}$ at $z = 0$ (figure 14.10). In a layer surrounding $z = 0$ (specifically $-0.88 < z < 0.88$), B_z is negative.

We seek a solution of (2.3.10) for the special case $\tilde{k} = \alpha$:

$$\sigma^2\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \alpha^2\right)\hat{w} = B_z\alpha^2\hat{w}.$$

Try the suggested function $\hat{w} = \text{sech}^2\beta z$, and seek a value of β for which the equation is satisfied. Guessing $\beta = \alpha$, we find that the equation is satisfied and the growth rate is

$$\sigma = \sqrt{B_{z0}/3}.$$

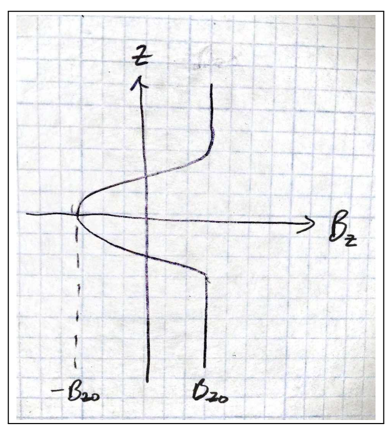

Figure 14.10: Sketch of the stratification profile for project 3.