# Chapter 8

# Instability in a rotating environment

## 8.1 Frontal zones

Imagine a fluid where buoyancy changes in the horizontal. Ordinarily, such a buoyancy distribution could not be sustained; the dense fluid would flow under the buoyant fluid, and the buoyant under the dense, until the buoyancy gradient became vertical. On a rotating planet, though, a horizontal buoyancy gradient can be maintained by the Coriolis force. One example is the atmospheric polar front, where cold polar air contrasts with warmer midlatitude air. Most major ocean currents have the same property; for example, the Gulf Stream carries warm water from the Gulf of Mexico into the cold North Atlantic.

The resulting equilibrium state is often unstable: the Coriolis force can maintain the density distribution in the mean, but any small disturbance will upset the balance. Midlatitude weather systems result from instabilities of the polar front, while the Gulf Stream continually spins off mesoscale eddies. Those instabilities are the focus of this chapter. [1]

The Coriolis acceleration pulls winds and currents to the right in the northern hemisphere, to the left in the southern. Figure 8.1 describes a frontal zone. The domain of interest covers only a small part of the front, so that the buoyancy varies smoothly. The geographical orientation of the $x$ and $y$ coordinates is arbitrary, but for definiteness we'll imagine that the view is to the west, with south at the left.

## 8.2 The equilibrium state: thermal wind balance

For this discussion we'll neglect viscosity and diffusion, but retain buoyancy and re-introduce the Coriolis acceleration:

$$\vec{\nabla}\cdot\vec{u} = 0 \tag{8.2.1}$$

$$\frac{D\vec{u}}{Dt} = -\vec{\nabla}\pi + b\hat{e}^{(z)} + \vec{u}\times f\hat{e}^{(z)} \tag{8.2.2}$$

$$\frac{Db}{Dt} = 0. \tag{8.2.3}$$

[1] The eddies that result from these instabilities can themselves manifest very similar instabilities. Like the unstable fronts that spawn them, eddies contain horizontal density gradients. These are maintained not by the Coriolis force but by the centrifugal force of the eddy's rotation. **That case is discussed elsewhere.**

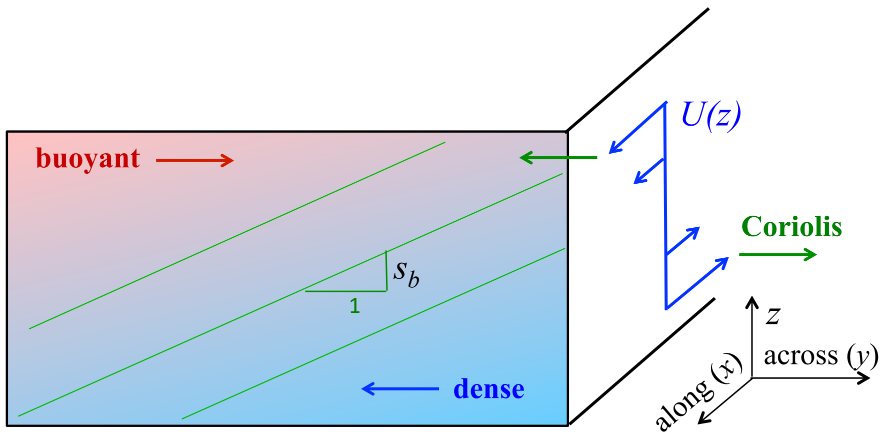


**Figure 8.1:** A baroclinic frontal zone in thermal wind balance. The buoyancy contrast creates a pressure gradient that is balanced by the Coriolis force. The isopycnal slope is $s_b$. Coordinates indicate the along-front ($x$) and across-front ($y$) directions and the vertical, $z$. The figure assumes that we're in the northern hemisphere and the Coriolis acceleration is therefore to the right.

To describe a baroclinic frontal zone like that shown in figure 8.1, we seek an equilibrium state in which buoyancy is a function of $y$ as well as $z$, and the velocity is purely zonal:

$$b = B(y,z)\ ; \quad \vec{u} = U\hat{e}^{(x)}.$$

No assumption is made about the spatial variation of $U$, but (8.2.1) requires $\partial U/\partial x = 0$, hence $U = U(y,z)$. The buoyancy equation (8.2.3) is satisfied automatically because there is no $x$-dependence for the zonal current to advect. For the same reason, the left-hand side of the momentum equation (8.2.2) is zero. The three components of (8.2.2) are then

$$\frac{\partial \Pi}{\partial x} = 0 \tag{8.2.4}$$
$$\frac{\partial \Pi}{\partial y} = -fU \tag{8.2.5}$$
$$\frac{\partial \Pi}{\partial z} = B. \tag{8.2.6}$$

In order, these equations tell us that:

- The pressure can depend only on $y$ and $z$.
- The meridional pressure gradient is balanced by the Coriolis force, i.e. the pressure is in geostrophic (literally "Earth turning" ) balance with the current.
- The vertical pressure gradient is in hydrostatic balance with the buoyancy.

Eliminating $\Pi$ between (8.2.5) and (8.2.6), we obtain the "thermal wind" balance[2]:

$$\boxed{f\frac{\partial U}{\partial z} = -\frac{\partial B}{\partial y}.} \tag{8.2.7}$$

[2] The name reflects the meteorological origins of the concept, but it is equally relevant in any rotating, stratified fluid

The horizontal variation of buoyancy is referred to as "baroclinicity". A useful measure of the strength of the baroclinicity is the isopycnal slope:

$$s_b = -\frac{\partial B/\partial y}{\partial B/\partial z},$$

the slope of a surface on which buoyancy is uniform (figure 8.1). Using (8.2.7), we can also write

$$s_b = \frac{f\partial U/\partial z}{\partial B/\partial z}.$$

## 8.3 The perturbation equations

We now linearize (8.2.1-8.2.3) by applying perturbations $\vec{u}'$, $\pi'$ and $b'$. As always, the perturbation is incompressible:

$$\vec{\nabla}\cdot\vec{u}' = 0. \tag{8.3.1}$$

The linearized momentum equation is

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\vec{u}' + \underbrace{\frac{\partial U}{\partial y}v'\hat{e}^{(x)}}_{(1)} + \frac{\partial U}{\partial z}w'\hat{e}^{(x)} = -\vec{\nabla}\pi' + b'\hat{e}^{(z)} + \underbrace{\vec{u}'\times f\hat{e}^{(z)}}_{(2)}, \tag{8.3.2}$$

while the buoyancy equation becomes,

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)b' + \underbrace{\frac{\partial B}{\partial y}v'}_{(3)} + \frac{\partial B}{\partial z}w' = 0, \tag{8.3.3}$$

Braced terms in (8.3.2) and (8.3.3) have not appeared in previous models (e.g. 5.3.5, 5.3.7) where there was no ambient rotation and the mean state varied only with $z$. The final term in (8.3.2), marked (2), is the Coriolis acceleration, and can be expanded as $fv'\hat{e}^{(x)} - fu'\hat{e}^{(y)}$. The remaining new terms, (1) and (3), represent advection of the horizontal gradients of $U$ and $B$ by the cross-front velocity perturbation $v'$.

## 8.4 Perturbation kinetic energy

As in section 3.9.1, we derive the equation for the perturbation kinetic energy by dotting $\vec{u}'$ onto the perturbation velocity equation which, in this case, is (8.3.2). The result is

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{|\vec{u}'|^2}{2} + \frac{\partial U}{\partial y}u'v' + \frac{\partial U}{\partial z}u'w' = -\vec{u}'\cdot\vec{\nabla}\pi' + b'w' + \vec{u}'\cdot(\vec{u}'\times f\hat{e}^{(z)}). \tag{8.4.1}$$

The final term vanishes, because the cross product is perpendicular to $\vec{u}'$. In physical terms, this reflects the fact that the Coriolis acceleration acts at right angles to the flow (e.g. to the right in the northern hemisphere). It therefore affects the *direction* of the flow but not the *magnitude*. The kinetic energy, being a measure of the magnitude, is unaffected by the Coriolis acceleration.

Rotation therefore has this property in common with viscosity (section 6.10): it cannot, by itself, transfer energy to the perturbation. It can, however, alter the form of the perturbation such that it gains energy via the shear or buoyancy production mechanisms.

We next use (8.2.1) to convert $\vec{u}'\cdot\vec{\nabla}\pi'$ to $\vec{\nabla}\cdot(\vec{u}'\pi')$ and bring the second and third terms to the right-hand side:

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\frac{|\vec{u}'|^2}{2}=-\vec{\nabla}\cdot(\vec{u}'\pi')-\frac{\partial U}{\partial y}u'v'-\frac{\partial U}{\partial z}u'w'+b'w'. \tag{8.4.2}$$

Applying a horizontal average and rearranging a little, we arrive at

$$\boxed{\frac{\partial}{\partial t}\frac{\overline{|\vec{u}'|^2}}{2}=-\frac{\partial}{\partial z}\overline{w'\pi'}-\frac{\partial U}{\partial z}\overline{u'w'}+\overline{b'w'}-\frac{\partial U}{\partial y}\overline{u'v'}.} \tag{8.4.3}$$

Most of this should look very familiar. The first and second terms on the right hand side were described in section 3.9.1: the first is the convergence of the vertical energy flux, which vanishes when integrated in the vertical; the second is the shear production.

The third term is the buoyancy flux that we discussed (in normal mode form) in section 5.11: it is positive (i.e. adding to the kinetic energy of the perturbation) if buoyant fluid is rising and dense fluid is falling. In the reverse case, the perturbation must do work against gravity to grow, so this term exerts a damping influence.

The fourth term is new, but it will be easily recognized as a second shear production term. Through it, the instability can exchange energy with the horizontal shear $\partial U/\partial y$.

## 8.5 The vertical vorticity equation

The planet's rotation represents a vorticity whose vertical component is $f$. Measured in an inertial reference frame, the total vorticity would be that of the flow as we measure it plus the extra contribution from the planet, a circumstance which affects the flow profoundly. In this chapter we will pay special attention to factors affecting vorticity. Since we have made the $f$-plane approximation ($f$=const.), the vertical component of vorticity is the most relevant.

The perturbation vertical vorticity is given by

$$\xi'=\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y} \tag{8.5.1}$$

We derive an evolution equation for $\xi'$ from the $x$ and $y$ components of the perturbation momentum equation (8.3.2), namely:

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)u'+\frac{\partial U}{\partial y}v'+\frac{\partial U}{\partial z}w'=-\frac{\partial \pi'}{\partial x}+fv' \tag{8.5.2}$$

and

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)v'=-\frac{\partial \pi'}{\partial y}-fu'. \tag{8.5.3}$$

Subtracting the $y$ derivative of (8.5.2) from the $x$ derivative of (8.5.3), we obtain

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\xi'=\underbrace{\frac{\partial^2 U}{\partial y^2}v'+\frac{\partial^2 U}{\partial y\partial z}w'}_{advection}+\underbrace{\frac{\partial U}{\partial z}\frac{\partial w'}{\partial y}}_{tilting}+\underbrace{f_a\frac{\partial w'}{\partial z}}_{stretching}. \tag{8.5.4}$$

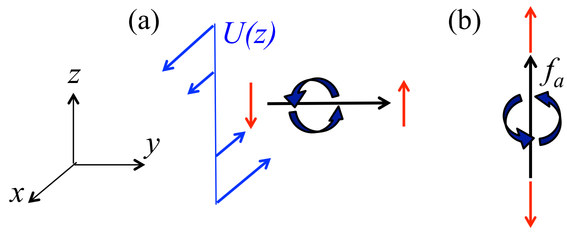


**Figure 8.2:** Mechanisms governing the perturbation vorticity as in (8.5.4). Red arrows depict the vertical velocity $w'$. (a) Tilting of the mean vertical shear by cross-front variations in $w'$. (b) Stretching of the absolute vertical vorticity by the vertical strain $\hat{w}'_z$.

The first two terms on the right hand side simply describe advection of the background vorticity $-\partial U/\partial y$ by the velocity perturbations $v'$ and $w'$. The third represents vortex tilting as shown in figure 8.2a. The vertical shear of the mean flow carries $y$-vorticity which is tilted toward the vertical by differences in vertical motion. The final term (figure 8.2b) represents vortex stretching. The total vertical vorticity (that of the planet plus the mean flow, as would be measured by an observed in outer space) is written as

$$f_a = f - \frac{\partial U}{\partial y}.$$

In the final term of (8.5.4) that total vorticity is stretched or compressed by the vertical strain $w'_z$.

## 8.6 Numerical solution method

Numerical solution of (8.3.1 -8.3.3) is more complicated than in the non-rotating cases we have studied previously for two main reasons: the pressure equation acquires a Coriolis term, and the variation of the background state with $y$ complicates the application of normal modes. Here we will describe those problems in turn, and then describe the solution.

### 8.6.1 The pressure equation

Taking the divergence of (8.3.2) gives the pressure equation

$$\nabla^2\pi' = -2U_y\frac{\partial v'}{\partial x} - 2U_z\frac{\partial w'}{\partial x} + \frac{\partial b'}{\partial z} + f\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right). \tag{8.6.1}$$

With $f \neq 0$, $u'$ and $v'$ appear on the right hand side, our usual tactic of using the pressure equation to eliminate those variables will have to be modified.

### 8.6.2 The normal mode approach and its limitations

The simplest approach to solving the perturbation equations (8.2.1, 8.3.2, 8.3.3) should now be familiar. We use normal mode solutions to convert these perturbation equations into a set of ordinary differential equations with independent variable $z$. This requires that the coefficients depend only on $z$, so that solutions of the form

$$u' = \hat{u}(z)e^{\sigma t}e^{\imath(kx+\ell y)}, \tag{8.6.2}$$

are valid. This places some constraints on the forms that the mean flow make take.

Consider the differentiated quantities $\partial U/\partial y$, $\partial U/\partial z$, $\partial B/\partial y$ and $\partial B/\partial z$. We require that (1) each of these derivatives be a function of $z$ only, and (2) that the thermal wind balance $\partial B/\partial y = -f\partial U/\partial z$ be satisfied. The most general forms for $U$ and $B$ satisfying these requirements are

$$U(y,z) = U_y y + U_z z, \tag{8.6.3}$$

and

$$B(y,z) = B_y y + \vartheta(z), \tag{8.6.4}$$

where $U_y$, $U_z$ and $B_y$ are constants, $B_y = -fU_z$, and $\vartheta$ is an arbitrary function of $z$.

Now consider the second terms in (8.3.2) and (8.3.3). These describe advection of the perturbation by the background flow, and have the form $U$ times $\partial/\partial x$ of something. Suppose first that the perturbation is independent of $x$. In that special case, the advection terms vanish regardless of the form of $U$. We will consider that case separately in section 8.7. For a general perturbation, the factor $U$ must depend only on $z$, i.e.

$$U_y = 0. \tag{8.6.5}$$

### 8.6.3 Numerical solution for normal modes

With the constraints (8.6.4) and (8.6.5) satisfied, we substitute (8.6.2) into (8.2.1 - 8.3.3) to obtain the normal mode perturbation equations

$$\begin{aligned}
\imath k\hat{u} + \imath\ell\hat{v} + \hat{w}_z &= 0 && (8.6.6)\\
(\sigma + \imath kU)\hat{u} + \imath kU_z\hat{w} &= -\imath k\hat{\pi} + f\hat{v} && (8.6.7)\\
(\sigma + \imath kU)\hat{v} &= -\imath\ell\hat{\pi} - f\hat{u} && (8.6.8)\\
(\sigma + \imath kU)\hat{w} &= -\hat{\pi}_z + \hat{b} && (8.6.9)\\
(\sigma + \imath kU)\hat{b} + B_y\hat{v} + \vartheta_z\hat{w} &= 0. && (8.6.10)
\end{aligned}$$

Instead of using $\hat{u}$ and $\hat{v}$, we define new variables

$$\hat{\xi} = \imath k\hat{v} - \imath\ell\hat{u}\,; \quad \hat{\chi} = \imath k\hat{u} + \imath\ell\hat{v}. \tag{8.6.11}$$

The first of these is just the normal mode form of the perturbation vertical vorticity defined in section 8.5. Its evolution equation, gotten by cross-differentiating (8.6.7) and (8.6.8), is

$$(\sigma + \imath kU)\hat{\xi} - \imath\ell U_z\hat{w} = f\hat{w}_z. \tag{8.6.12}$$

This is the normal mode form of (8.5.4). (The limitation $U_y = 0$ has been imposed, so that $y$-derivatives of $U$ vanish and $f_a = f$.)

Our next goal is to eliminate $\hat{\pi}$ from the vertical momentum equation (8.6.9). The troublesome pressure equation (8.6.1) can be written as

$$\nabla^2\hat{\pi} = -2\imath kU_z\hat{w} + \hat{b}_z + f\hat{\xi} \tag{8.6.13}$$

where, as usual,

$$\nabla^2 = \frac{d^2}{dz^2} - \tilde{k}^2\,, \text{ and } \tilde{k} = \sqrt{k^2 + \ell^2}.$$

In section 3.2.2, we discussed in detail the procedure of eliminating pressure by (1) deriving a Helmholtz equation like the one above, (2) taking the Laplacian of the vertical velocity equation, and combining the the two. Thanks to our use of $\hat{\xi}$, that procedure will work here. Starting with (8.6.9) and applying $\nabla^2$, we obtain

$$(\sigma + \imath kU)\nabla^2\hat{w} + 2\imath kU_z\hat{w}_z = -\nabla^2\hat{\pi}_z + \nabla^2\hat{b},$$

remembering that $\partial^2 U/\partial z^2 = 0$. Differentiating (8.6.13) and substituting leads, after some gratifying cancellations, to

$$(\sigma + \imath kU)\nabla^2\hat{w} = -\tilde{k}^2\hat{b} - f\hat{\xi}_z. \tag{8.6.14}$$

In (8.6.12) and (8.6.14) we have two equations for the three unknown functions $\hat{\xi}$, $\hat{w}$ and $\hat{b}$. We will add the buoyancy equation (8.6.10), but note that it involves dependence on $\hat{v}$. To remove this dependence, we solve the pair of equations (8.6.11) for $\hat{v}$:

$$\hat{v} = -\frac{\imath k}{\tilde{k}^2}\hat{\xi} - \frac{\imath\ell}{\tilde{k}^2}\hat{\chi},$$

and remember that $\hat{\chi} = -\hat{w}_z$ by continuity. With this substitution, (8.6.10) becomes

$$(\sigma + \imath kU)\tilde{k}^2\hat{b} = \imath kB_y\hat{\xi} - \imath\ell B_y\hat{w}_z - \tilde{k}^2\vartheta_z\hat{w}. \tag{8.6.15}$$

> *Exercise:* Compare (8.6.14, 8.6.15) with (5.3.11, 5.3.12) and also with (7.1.15, 7.1.16). Note all differences, and make sure you can explain each in terms of the different assumptions that have been made.

We now have three equations, (8.6.12), (8.6.14) and (8.6.15), for the threee unknowns $\hat{\xi}$, $\hat{w}$ and $\hat{b}$. These can be solved using the matrix method as in previous examples, albeit in a slightly more complicated form. We write (8.6.12), (8.6.14) and (8.6.15) as a matrix differential equation:

$$\sigma\begin{pmatrix} 1 & 0 & 0 \\ 0 & \nabla^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{\xi} \\ \hat{w} \\ \tilde{k}^2\hat{b} \end{pmatrix} = \begin{pmatrix} -\imath kU & \imath\ell U_z + fD^{(1w)} & 0 \\ -fD^{(1\xi)} & -\imath kU\nabla^2 & -1 \\ \imath kB_y & -\imath\ell B_y D^{(1w)} - \tilde{k}^2\vartheta_z & -\imath kU \end{pmatrix}\begin{pmatrix} \hat{\xi} \\ \hat{w} \\ \tilde{k}^2\hat{b} \end{pmatrix}, \tag{8.6.16}$$

with

$$\nabla^2 = D^{(2)} - \tilde{k}^2$$

and

$$D^{(1)} = \frac{d}{dz}\ ;\ D^{(2)} = \frac{d^2}{dz^2}.$$

The first derivative operators in (8.6.16) are marked with the superscripts $w$ and $\xi$ to indicate that their matrix equivalents may be different depending on the choice of boundary conditions (see below). We now replace the derivative operators $D^{(1)}$ and $D^{(2)}$ with derivative matrices as defined previously (sections 1.5.2, 3.6, 6.5 and 7.2), remembering to incorporate the appropriate boundary conditions. The result is a generalized eigenvalue problem with $3N \times 3N$ matrices:

$$\sigma\,\underset{\sim}{A}\vec{x} = \underset{\sim}{B}\vec{x}.$$

The eigenvector $\vec{x}$ is a concatenation of the discretized forms of $\hat{\xi}$, $\hat{w}$ and $\tilde{k}^2\hat{b}$. (The factor $\tilde{k}^2$ is absorbed into the buoyancy eigenfunction only because it makes (8.6.16) a bit tidier.)

### 8.6.4 Boundary conditions

At this point we have explored numerous options for boundary conditions. For $\hat{w}$, we use the impermeable boundary $\hat{w}=0$. For buoyancy, (8.6.16) does not require a boundary condition since $\hat{b}$ is not differentiated.

The new variable is the vertical vorticity $\xi$. Differentiating the definition (8.5.1), we have

$$\frac{\partial \xi'}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right) = \frac{\partial}{\partial x}\frac{\partial v'}{\partial z} - \frac{\partial}{\partial y}\frac{\partial u'}{\partial z}. \tag{8.6.17}$$

If we now assume that the boundary is *frictionless*, as described in section 6.4.2, then this reduces to $\partial \xi'/\partial z = 0$ or, in normal mode form,

$$\hat{\xi}_z = 0.$$

One should therefore design the derivative matrices $\underset{\sim}{D}^{(2)}$ and $\underset{\sim}{D}^{(1w)}$ to be consistent with $\hat{w}=0$, and $\underset{\sim}{D}^{(1\xi)}$ to give $\hat{\xi}_z = 0$ (i.e. make the top and bottom rows all zeroes).

### 8.6.5 Shear scaling

Suppose that (8.6.16) has a solution algorithm:

$$[\sigma, \hat{\xi}, \hat{w}, \hat{b}] = \mathscr{F}(z, U_z, \vartheta_z, f;\ k, \ell).$$

Now let us choose the time scale to be $1/U_z$ and our length scale to be the domain height $H$. The nondimensionalization is straightforward. Scalings of particular interest are

$$f^* = \frac{f}{U_z},$$

and

$$\vartheta^*_{z^*} = \frac{\vartheta_z}{U_z^2} = Ri(z),$$

the gradient Richardson number. Note that $Ri$ can vary with height.

The scaled equations are isomorphic to (8.6.16) , and therefore

$$[\sigma^*, \hat{\xi}^*, \hat{w}^*, \hat{b}^*] = \mathscr{F}(z^*, 1, Ri(z^*), f^*;\ k^*, \ell^*).$$

Other time scales are possible, e.g. $1/f$. If the stratification $\vartheta_z$ has an identifiable "characteristic" value $B_z$, then $1/\sqrt{B_z}$ is also a viable time scale.

## 8.7 Analytical solution 1: symmetric instability

As noted in section 8.6.2, the perturbation equations (8.3.1, 8.3.2, 8.3.3) have normal mode solutions of the form (8.6.7) *only* if one of two conditions holds:

- the mean current $U$ is a function of $z$ only, or

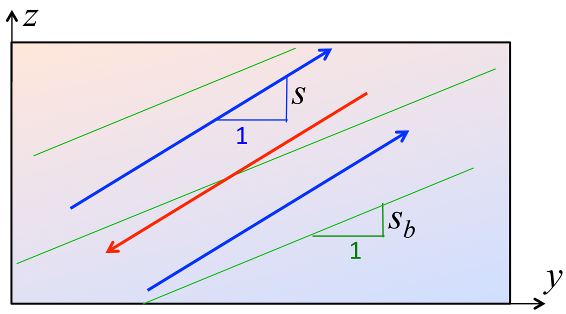


**Figure 8.3:** Definition sketch for symmetric instability. The thin green lines are isopycnals, with slope $s_b = -B_y/B_z$. Arrows show buoyant (red) and dense (blue) currents whose slope is $s = -\ell/m$. The flow is invariant in the along-front ($x$) direction.

- the perturbation does not vary in the $x$ direction, so that the advection operator $U\partial/\partial x$ is zero whatever the form of $U$.

In the latter case, the flow is *symmetric* in the sense that nothing varies in the $x$ direction. The class of unstable modes having this property is termed symmetric instability (figure 8.3). In this case, the mean flow (8.6.5) can be replaced by (8.6.3), which allows for a uniform shear in the cross-front ($y$) direction.

For this discussion we make two additional simplifying assumptions:

- To allow an analytical solution, the vertical buoyancy gradient $\vartheta_z(z)$ is assumed to be uniform, so the mean buoyancy is

$$B = B_y y + B_z z,$$

  where $B_y$ and $B_z$ are constants.

- The perturbation is *quasi-hydrostatic* in the sense that $b'$ can be approximated by $\partial\pi'/\partial z$.[3] This assumption is easily relaxed; it merely simplifies the algebra.

With these assumptions, the coefficients of the perturbation equations (8.2.1,8.3.2, 8.3.3) are all constants, and we can seek a solution using normal modes of the "plane wave" form

$$u' = \hat{u}e^{\sigma t}e^{\imath(\ell y+mz)},$$

where the complex amplitude $\hat{u}$ is a constant.

The normal mode equations are now

$$\begin{aligned}
\imath\ell\hat{v} + im\hat{w} &= 0 && (8.7.1)\\
\sigma\hat{u} + U_y\hat{v} + U_z\hat{w} &= f\hat{v} && (8.7.2)\\
\sigma\hat{v} &= -\imath\ell\hat{\pi} - f\hat{u} && (8.7.3)\\
\imath m\hat{\pi} &= \hat{b} && (8.7.4)\\
\sigma\hat{b} + B_y\hat{v} + B_z\hat{w} &= 0. && (8.7.5)
\end{aligned}$$

[3] Admonition: This does *not* mean that $\hat{w} = 0$, as a look at the vertical momentum equation,

$$(\sigma + \imath kU)\hat{w} = -\hat{\pi}_z + \hat{b},$$

might suggest (cf. 8.6.9). Vertical motions, slow as they may be, play a critical role in the other equations by advecting the vertical gradients $U_z$ and $B_z$. The assumption we make here is only that $\hat{w}$, as it appears in (8.6.9), is a small difference between two large quantities $\hat{b}$ and $\hat{\pi}_z$, so we can approximate the two large quantities as equal.

We now solve (8.7.1) for $\hat{w}$ and (8.7.4) for $\hat{\pi}$, leaving three homogeneous equations for the three unknowns $\hat{u}$, $\hat{v}$ and $\hat{b}$. The characteristic equation is

$$\sigma^2 = \frac{\ell^2}{m^2}B_z - 2\frac{\ell}{m}fU_z - f(f - U_y), \tag{8.7.6}$$

where use has been made of the thermal wind relation $B_y = -fU_z$ (cf. 8.2.7).

A few abbreviations will be useful here. First, note that the wave vector components $\ell$ and $m$ appear only as the ratio $\ell/m$, which is minus the ratio $\hat{w}/\hat{v}$ according to (8.7.1). As shown in figure 8.3, this is the slope of the planes to which motion is restricted in the normal mode solution (8.7):

$$s = -\frac{\ell}{m}.$$

The slope of the isopycnals (thin lines in figure 8.3) is

$$s_b = -\frac{B_y}{B_z} = \frac{fU_z}{B_z}.$$

Finally, the quantity $f - U_y$ is the absolute vorticity $f_a$, as defined in section 8.5. With these abbreviations the characteristic equation (8.7.6) becomes

$$\boxed{\sigma^2 = -B_z s^2 + 2B_z s_b s - f f_a.} \tag{8.7.7}$$

Differentiation with respect to $s$ shows that the growth rate is a maximum when

$$s = s_b,$$

and that

$$\sigma_{max} = |f|\left(\frac{U_z^2}{B_z} - \frac{f_a}{f}\right)^{1/2}. \tag{8.7.8}$$

Remembering that $B_z/U_z^2$ is the Richardson number $Ri$, the condition for $\sigma_{max}$ to be real can be written as

$$Ri < \frac{f}{f_a}. \tag{8.7.9}$$

For example, if there is no across-front shear, then $f_a = f$ and the condition for stability is $Ri < 1$.

- Unlike the similar condition $Ri < 1/4$ for shear instability, (8.7.9) is a necessary *and sufficient* condition, i.e. instability is guaranteed if the condition is satisfied.
- Like convection in an inviscid, unbounded fluid, symmetric instability has no preferred length scale (cf. section 2.3.2). The growth rate depends only on the orientation of the wave vector.
- The across-front shear $U_y$ can have either a stabilizing or a destabilizing influence depending on its sign. If $U_y$ has the same sign as $f$, (i.e. $U_y > 0$ in the northern hemisphere), the across-front shear reduces the absolute vorticity $f_a$. From (8.7.8) we see that this increases the maximum growth rate, while (8.7.9) shows that the criterion for $Ri$ is relaxed.
- When $s > s_b$, as in figure 8.3, the instability does work against gravity, whereas the opposite is true if $s < s_b$. For the fastest-growing symmetric instability, motion is along isopycnals ($s = s_b$) and the mode exchanges no energy with the gravitational field. The energy source must therefore be the shear of the along-front current.

## 8.8 Analytical solution 2: baroclinic instability

For perturbations that vary in the alongfront direction (as distinct from symmetric instability), analytical solutions are available provided that we make the set of simplifying assumptions that define a quasigeostrophic flow. The essential assumption is that the flow varies slowly relative to the Earth's rotation, i.e. on a time scale much greater than a day. The conditions for quasigeostrophy to hold are described in much more detail elsewhere, e.g. **Pedlosky (1987)**. Here we will give only enough detail to make the approximation plausible. We will find, however, that the characteristics of the predicted instability correspond well with those of (1) midlatitude storms and (2) oceanic mesoscale eddies.

### 8.8.1 The quasigeostrophic potential vorticity perturbation

For simplicity we assume that (8.6.3) holds, and we hereafter write partial derivatives as subscripts. The vertical component of the perturbation vorticity, introduced in section 8.5, is

$$\xi' = v'_x - u'_y. \tag{8.8.1}$$

Its evolution equation (8.5.4) is now

$$\xi'_t = -U\xi'_x + U_z w'_y + f_a w'_z. \tag{8.8.2}$$

We now make two critical assumptions. First, we assume that the vortex tilting effect is negligible in comparison with the stretching effect, i.e. $|U_z w'_y| \ll |f_a w'_z|$ (figure 8.2). Second, we neglect $U_y$ relative to $f$ so that $f_a$ is replaced by $f$.

This leaves us with the approximate vertical vorticity equation

$$\xi'_t + U\xi'_x = f w'_z. \tag{8.8.3}$$

This is a single equation for two unknowns, so we need more information (i.e. assumptions) to make a complete theory. The strategy is to approximate both $\xi'$ and $\hat{w}'_z$ in terms of the pressure perturbation $\pi'$.

*The left hand side*

We begin with an assumption concerning the perturbation momentum equations (8.5.2) and (8.5.3). In each of those equations, is is often true that the terms on the right hand side, i.e. the pressure gradient and the Coriolis acceleration, are large in magnitude compared with the terms on the left hand side. In other words, the left hand side is not zero, but it is a small difference of large numbers. It follows that the pressure gradient and Coriolis terms must be nearly equal. If those terms are, in fact, equal, the perturbation is in geostrophic balance, and the horizontal velocity components can be represented entirely in terms of the pressure perturbation:

$$u^{(g)} = -\frac{\hat{\pi}'_y}{f}\,; \quad v^{(g)} = \frac{\hat{\pi}'_x}{f}. \tag{8.8.4}$$

In fact, the horizontal velocity perturbation can be thought of as a geostrophic part and an ageostrophic part:

$$u' = u^{(g)} + u^{(a)}\,; \quad v' = v^{(g)} + v^{(a)}.$$

We can do the same with the perturbation vorticity:

$$\xi' = \xi^{(g)} + \xi^{(a)}$$

where

$$\xi^{(g)} = v_x^{(g)} - u_y^{(g)} = \frac{1}{f}\nabla_H^2 \pi'.$$

Our assumption is that $|\xi^{(a)}| \ll |\xi^{(g)}|$, so that

$$\xi' = \frac{1}{f}\nabla_H^2 \pi'. \tag{8.8.5}$$

*The right hand side*

Suppose we try to approximate $w'_z$ in the same way, by assuming that it is dominated by its geostrophic part. From continuity we have

$$\begin{aligned}
w'_z = -(u'_x + v'_y) &= -(u_x^{(g)} + u_x^{(a)} + v_y^{(g)} + v_y^{(a)}) \\
&= -\left(-\frac{\pi'_{yx}}{f} + u_x^{(a)} + \frac{\pi'_{xy}}{f} + v_y^{(a)}\right) \\
&= -(u_x^{(a)} + v_y^{(a)})
\end{aligned}$$

So, we can't assume that the ageostrophic part of $w'_z$ is negligible, as we did with $\xi'$, because it's the only part. Instead, we invoke the buoyancy equation (8.3.3), which we write in the form

$$B_z w' = -B_y v' - b'_t - U b'_x. \tag{8.8.6}$$

We assume once again that $v'$ is dominated by its geostrophic part $v^{(g)}$, and moreover that the perturbation is in hydrostatic balance $b' = \pi'_z$. Making the appropriate substitutions, (8.8.6) becomes

$$B_z w' = -\frac{B_y}{f}\pi'_x - \pi'_{zt} - U\pi'_{zx}. \tag{8.8.7}$$

and, after a differentiation,

$$B_z w'_z = -\frac{B_y}{f}\pi'_{xz} - \pi'_{zzt} - U\pi'_{zzx} - U_z\pi'_{zx}.$$

Remembering the thermal wind balance $fU_z = -B_y$, we see that the first and last terms on the right hand side cancel, and therefore

$$w'_z = -\frac{1}{B_z}(\pi'_{zzt} - U\pi'_{zzx}). \tag{8.8.8}$$

*The potential vorticity equation*

Inserting (8.8.8) and (8.8.5) into (8.8.3), we have

$$q'_t + U q'_x = 0, \tag{8.8.9}$$

where $q'$ is the linearized, quasigeostrophic potential vorticity

$$\boxed{q' = \nabla_H^2 \pi' + \frac{f^2}{B_z}\pi'_{zz}.} \tag{8.8.10}$$

Equation (8.8.9) states that, to an observer moving with the mean along-front current $U$, the potential vorticity of the perturbation remains constant. If the perturbation is in fact growing exponentially, then its potential vorticity can only have one value and that is zero.

We next reconfigure (8.8.9,8.8.9) so that it can be solved in a finite vertical domain with impermeable upper and lower boundaries. We assume that $\pi'$ has the normal mode form

$$\pi' = \hat{\pi}(z)e^{\imath k(x-ct)+\imath \ell y},$$

and similarly for $q'$. Equation (8.8.9) becomes

$$\imath k(U-c)\hat{q} = 0$$

or, as anticipated, $\hat{q} = 0$ for $\sigma_r \neq 0$. From (8.8.10) we now obtain an ordinary differential equation

$$\boxed{\hat{\pi}_{zz} - \mu^2 \hat{\pi} = 0,} \tag{8.8.11}$$

where

$$\mu = \frac{\tilde{k}}{P} \tag{8.8.12}$$

is a scaled vertical wavenumber. The Prandtl ratio $P$ is defined as

$$P = \frac{|f|}{\sqrt{B_z}}, \quad \text{or } P = \frac{1}{Ro\sqrt{Ri}}, \tag{8.8.13}$$

and $\tilde{k}^2 = k^2 + \ell^2$ as usual.

*The impermeable boundary condition*

We must now express the boundary condition $\hat{w} = 0$ in terms of $\hat{\pi}$. To this end we write (8.8.7) in the normal mode form

$$B_z \hat{w} = -\frac{B_y}{f}\imath k\hat{\pi} - \imath k(U-c)\hat{\pi}_z.$$

or, invoking thermal wind balance $B_y = -fU_z$,

$$B_z \hat{w} = \imath k[U_z\, \hat{\pi} - (U-c)\hat{\pi}_z].$$

The boundary condition $\hat{w} = 0$ is therefore equivalent to

$$\boxed{(U-c)\hat{\pi}_z - U_z \hat{\pi} = 0.} \tag{8.8.14}$$

### 8.8.2 Eady waves

We can solve (8.8.11) with (8.8.14) imposed at upper and lower boundaries, or with one boundary at infinity. In the latter case, the solution describes Eady waves, which are analogous to the vortical waves discussed in section 3.13.1. And, like the vortical waves, a pair of Eady waves can resonate to drive exponential growth.

Before deriving the Eady wave dispersion mechanism, we give a qualitative description of the mechanism. In figure 8.4a, we are looking across a frontal zone from the warm (buoyant) side, shaded in red. Now consider an imaginary horizontal surface, and suppose that the fluid is displaced upward and downward by some means so that the surface varies sinusoidally in the along-front direction. At the top of the domain is an impenetrable horizontal boundary where the amplitude of the displacement must decrease to zero as shown by the thin curves.

Because the vertical motion drops to zero at the boundary, the fluid above each trough of the perturbation experiences an extensional strain, $w'_z > 0$, which stretches, and thereby amplifies, the ambient vorticity $f$.

The result is a clockwise increment of vorticity as shown at the top of figure 8.4b by the curved arrows. In contrast, the strain above each crest is compressive, effectively reducing the ambient vorticity, so that the increment is counterclockwise. Between these strips of oppositely-signed vorticity increments are cross-front currents which carry alternately dense (blue) and buoyant (red) fluid.

In each cross-front current, gravity acts to accelerate the fluid vertically: upward in buoyant currents, downward in dense currents (figure 8.4c). As a result, the nodes of the sinusoidal disturbance are displaced vertically such that the whole pattern propagates to the left.

The leftward propagation is relative to the mean along-front current, which must be rightward to maintain thermal wind balance. At the *lower* boundary, the same dynamic supports a right-going wave in a leftward background current. It is therefore possible for both waves to be stationary and, if their relative phases are such that the vertical motion of one wave reinforces the crests and troughs of the other, positive feedback leads to a growing instability.

To derive a dispersion relation for the Eady wave shown in figure 8.4, we solve (8.8.11), with (8.8.14) imposed at the upper boundary and the condition that the solution remain bounded as $z \to -\infty$. The general solution of (8.8.11) is

$$\hat{\pi} = \alpha e^{\mu z} + \beta e^{-\mu z}, \tag{8.8.15}$$

where $\alpha$ and $\beta$ are constants. Boundedness as $z \to -\infty$ requires that $\beta = 0$. In this case $\hat{\pi}_z = \mu \hat{\pi}$, and the boundary condition becomes

$$(U_u - c)\mu - U_z = 0,$$

where $U_u$ is the mean along-front velocity at the boundary. Solving for $c$, we have

$$c = U_u - \frac{U_z}{\mu}. \tag{8.8.16}$$

For a wave at a lower boundary, we carry out the same steps, this time requiring boundedness as $z \to +\infty$. In this case $\alpha = 0$, $\hat{w}_z = -\mu \hat{w}$, and the boundary condition gives the dispersion relation

$$c = U_l + \frac{U_z}{\mu}, \tag{8.8.17}$$

where $U_l$ is the mean along-front velocity at the lower boundary.

The upper and lower waves propagate oppositely relative to the mean flow at their respective boundaries. It is therefore possible that their phase speeds could be equal. In that case, the two waves might resonate and drive exponential growth. To see if this possibility is in fact true, we must solve (8.8.11) with (8.8.14) imposed at both the upper and lower boundaries.

### 8.8.3 The Eady mode of baroclinic instability

We now consider a finite domain, with coordinates chosen such that boundaries are at $z = \pm H/2$ and the mean along-front velocity is $U = U_z z$. The general solution of (8.8.15) must satisfy the boundary conditions

$$\left(-\frac{U_z H}{2} - c\right)\hat{\pi}_z + U_z \hat{\pi} = 0 \quad \text{at } z = -\frac{H}{2}$$

and

$$\left(\frac{U_z H}{2} - c\right)\hat{\pi}_z + U_z \hat{\pi} = 0 \quad \text{at } z = \frac{H}{2}$$

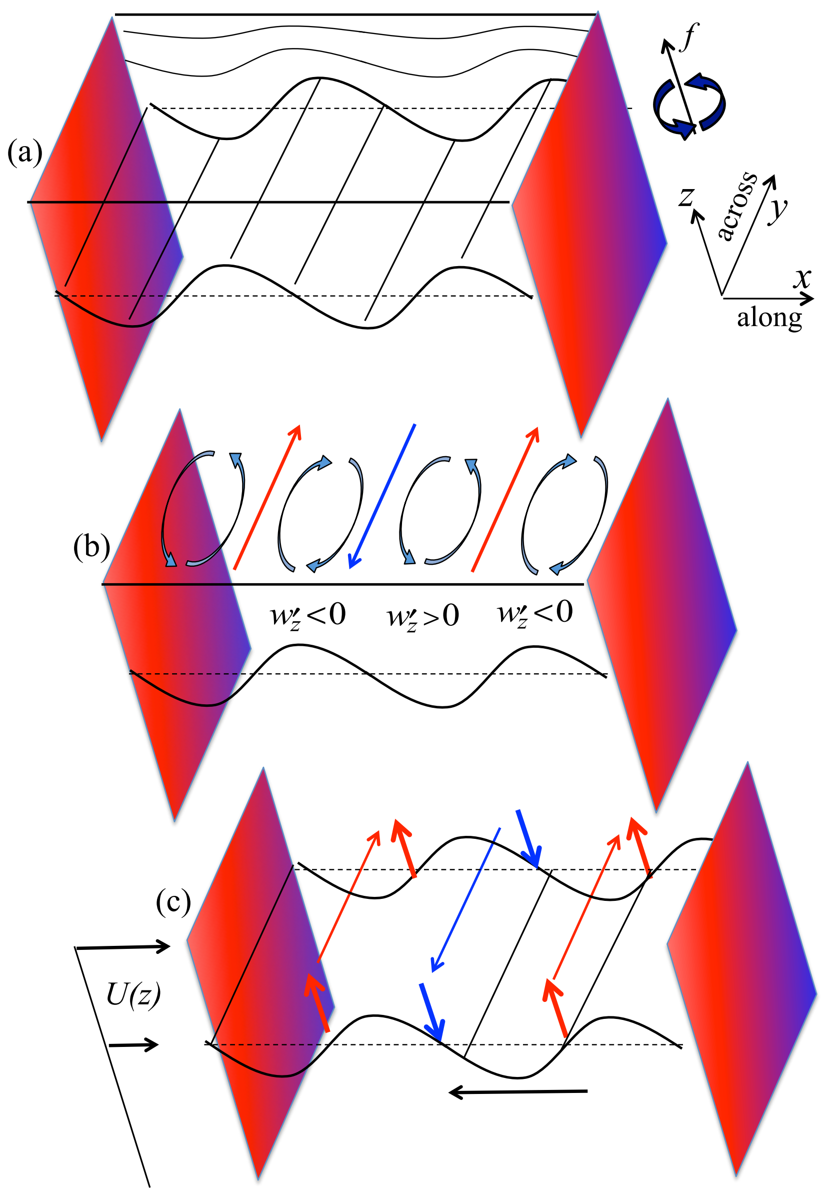


**Figure 8.4:** Mechanics of Eady wave propagation. (a) Sinusiodal perturbation of a frontal zone in thermal wind balance. Shading on the cross-sections indicates the mean buoyancy distribution: dense (blue) and buoyant (red). The figure assumes $f>0$, i.e. that we are in the northern hemisphere. (b) Circulations induced by stretching and compression of the ambient vorticity, and the resulting cross-front circulations. (c) Advected buoyancy drives rising and sinking motions which cause the pattern to propagate to the left, counter to the mean along-front flow $U$.

Substituting (8.8.15) and solving for $c$, we obtain

$$\boxed{c^{*2} = \frac{1}{4} - \frac{\coth\mu^*}{\mu^*} + \frac{1}{\mu^{*2}},} \tag{8.8.18}$$

where the nondimensional phase speed and vertical scale are

$$c^* = \frac{c}{U_z H}\,; \quad \mu^* = \mu H. \tag{8.8.19}$$

The solution (8.8.18) is shown in figure 8.5. For $\mu^* > 2.40$, $c$ is real and represents oppositely propagating modes. In the limit $\mu^* \to \infty$ (in which the domain height is large compared to the vertical scale $\mu^{-1}$), these correspond to the Eady waves described in section 8.8.2. [4]

As $\mu^*$ decreases from infinity, the two Eady waves become close in phase speed. At $\mu^* = 2.40$, both phase speeds reach zero, i.e. the waves become phase locked. For $\mu^* < 2.40$, $c$ is imaginary and we therefore have unstable modes with real growth rate $\sigma$. The growth rate is proportional to the product $\mu^* c_i^*$:

$$\sigma = kc_i \;=\; \frac{k}{\tilde{k}}\,\tilde{k}\,c_i^* U_z H \;=\; \frac{k}{\tilde{k}}\,\frac{\mu}{P}\,c_i^* U_z H \;=\; \frac{k}{\tilde{k}}\,\frac{U_z}{P}\,\mu^* c_i^*, \tag{8.8.20}$$

where (8.8.12) and (8.8.19) have been used.

Instead of oppositely-propagating waves, the solutions represent one growing and one decaying mode (figure 8.5). The product $\mu^* c_i^*$ reaches a maximum of 0.31 at $\mu^* = 1.61$ (circle on figure 8.5), then drops to zero at $\mu^* = 0$. This band of unstable modes $0 < \mu^* \le 2.40$ is the Eady mode of baroclinic instability, which we will call the Eady mode for short.

We will of course be most interested in the fastest growing Eady mode. The optimal nondimensional length scale $\mu^* = 1.61$ corresponds to

$$\tilde{k} = \frac{1.61}{H} P.$$

The wavelength is

$$\lambda = \frac{2\pi}{\tilde{k}} = \frac{3.9}{P} H.$$

This is sometimes written as

$$\lambda = 3.9 L_d,$$

where

$$L_d = \frac{H}{P}$$

is called the deformation radius.

[4] For $\mu^* \gg 1$, $\coth\mu^*$ rapidly approaches 1. Therefore,

$$c^{*2} \;\approx\; \frac{1}{4} - \frac{1}{\mu^*} + \frac{1}{\mu^{*2}} \;=\; \left(\frac{1}{2} - \frac{1}{\mu^{*2}}\right)^2.$$

In dimensional terms,

$$c \;\approx\; \pm\left(\frac{U_z H}{2} - \frac{U_z}{\mu}\right).$$

which is equivalent to the dispersion relations (8.8.16, 8.8.17) for the upper and lower Eady waves derived in section 8.8.2.

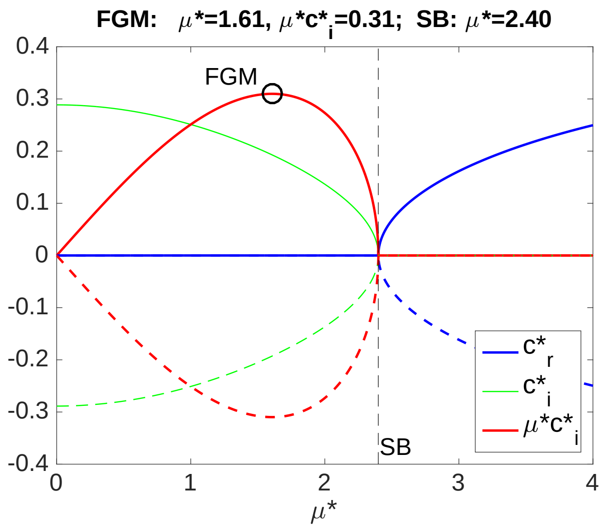


**Figure 8.5:** Phase speed (real=blue, imaginary=green) and scaled growth rate $\mu^* c_i^*$ (red) versus vertical length scale for the Eady model of baroclinic instability, based on (8.8.18). Scalings are defined by $\mu* = \mu H$ and $c* = c/U_z H$. Annotations mark the fastest-growing mode (FGM) and the stability boundary (SB).

For the optimal value of $\mu^*$, the dimensional growth rate is given by (8.8.20):

$$\sigma = 0.31\,\frac{k}{\tilde{k}}\,P\,U_z.$$

Like shear instability, the Eady mode grows fastest when $k = \tilde{k}$, or $\ell = 0$, i.e. when the wave vector is aligned with the mean along-front flow. The maximum growth rate is then proportional to the thermal wind shear and also to the Prandtl ratio:

$$\sigma \;=\; 0.31\,P\,U_z.$$

There are two interesting alternative ways to express this growth rate. First:

$$\sigma = 0.31\,\sqrt{B_z}\,s_b,$$

i.e. the growth rate depends on the strength of the stratification and the degree to which the isopycnals are tilted. Second:

$$\sigma = 0.31\,\frac{|f|}{\sqrt{Ri}}. \tag{8.8.21}$$

### 8.8.4 Typical terrestrial values

*The Prandtl ratio*

We now look at some typical dimensional values, beginning with the Prandtl ratio $f/\sqrt{B_z}$. A typical mid-latitude value for $f$ is $10^{-4}\mathrm{s}^{-1}$ (section 1.6). In the troposphere, a typical value for $B_z$ is $10^{-4}\mathrm{s}^{-2}$. Coincidentally, this value is also typical of the upper ocean. Therefore, the Prandtl ratio is near 0.01 in both fluids.

*The troposphere*

If we take $H$ to be the atmospheric scale height, which is near 10km, then the deformation radius is 1000km and the wavelength of the fastest-growing Eady mode is about 4000km. Now suppose that the wind speed changes by 10m/s over the scale height $H$, so that $U_z = 10^{-3}\text{s}^{-1}$. The growth rate is then $3 \times 10^{-6}\text{s}^{-1}$, for an e-folding time of 4 days. These estimates are "in the ballpark" for midlatitude weather systems, which develop over a few days and have longitudinal extents of a few thousand km.

*The thermocline*

In the ocean, the depth over which baroclinic instability acts is more like 1km, so the predicted wavelength is 1/10 of the atmospheric value, say 400km. This is a typical length scale for mesoscale eddies. If the current changes by 0.1m/s, then $U_z$ is $10^{-4}\text{s}^{-1}$, and the predicted e-folding time is about a month, not too different from the time scale for mesoscale eddy growth.

## 8.9 Summary

- A stratified fluid in a rotating environment is in equilibrium if the thermal wind balance holds: $fU_z = -B_y$.
- The perturbation equations can be solved numerically after the introduction of the vertical vorticity perturbation.
- Symmetric instability
  - requires $Ri < f_a/f$
  - does not vary in the along-front ($x$) direction
  - drives motions on isopycnals in the $(y-z)$ plane
  - has no preferred wavelength
- Baroclinic instability (the Eady model)
  - computed analytically under the quasigeostrophic approximation
  - motion does not vary in the cross-front ($y$) direction
  - motion is sinusoidal, stationary with respect to the central plane
  - wavelength $\lambda = 3.9H/P$, where $P = |f|/\sqrt{B_z}$.
  - growth rate $\sigma = 0.31PU_z = 0.31|f|/\sqrt{Ri}$.
  - can be understood as a resonance of waves

## 8.10 References

Pedlosky, J. 1987: *Geophysical fluid dynamics, 2nd ed.*, Springer-Verlag.