Chapter 5

Parallel shear flow: the effects of inhomogeneity

5.1 Introduction

The Earth's oceans and atmosphere tend to be density-stratified. The primary reason is heat radiation. At the equator, both ocean and atmosphere gain heat from the sun. Near the poles, the ocean loses heat to the atmosphere, while the atmosphere radiates heat into space at all latitudes. Ocean density is also governed by salinity. All of these processes create inhomogeneities which, under the action of gravity, tend to rearrange themselves into horizontal layers.

When layers move relative to one another, there is the likelihood of shear instability, as we have seen. But if the upper layer is more buoyant than the lower layer, then a growing perturbation must do work against gravity, and its growth is slowed as a result. If the buoyancy difference is sufficiently strong, the flow will be stable. This competition between shear U_z and stratification B_z is quantified using the gradient Richardson number $Ri = B_z/U_z^2$: if Ri is small, shear dominates; if Ri is large, stratification dominates. The value of Ri needed to stabilize a flow depends on many details, but is typically near 1/4.

In figure 5.1, observations at 550m depth off the Canary Islands show a cold bottom layer where the speed of the tidal current drops to zero. The shear between the bottom layer and the warmer overlying ocean overcomes the buoyancy difference, resulting in Kelvin-Helmholtz instability. The same instability occurs in air, as made visible by the fog layer in figure 5.2.

In this chapter we will explore stratification effects analytically, taking advantage of the relative simplicity of the inviscid equations. Numerical solution methods, oblique modes, scaling transformations and the energy budget will be taken up in the next chapter after we have incorporated viscosity and diffusion.

5.2 The Richardson number

In stably stratified shear flow, the growth of instability results from a competition between shear and stratification:

- $U_z \neq 0$ destabilizes

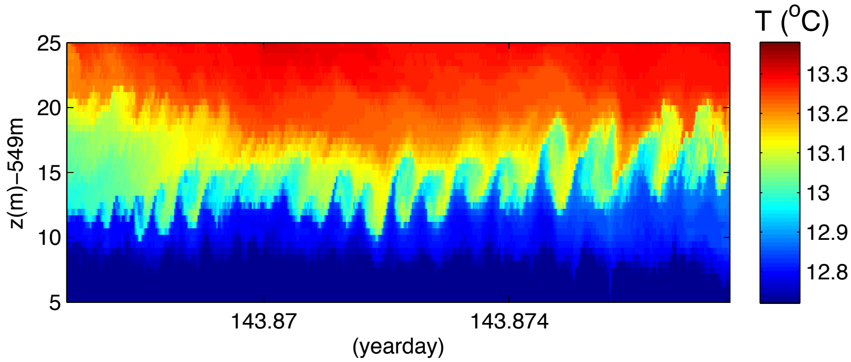


Figure 5.1: Temperature variations in a downslope tidal flow. This time series was constructed using data from multiple temperature sensors spaced vertically on a chain above the sea floor at 550 m depth. Typical wavelengths are inferred to be 75 m. Graphic courtesy H. van Haren (van Haren and Gostiaux, 2010)

Figure 5.2: Instability in a cold fog layer near Nares Strait in northern Canada. Photo by Scott McAuliffe, Oregon State University.

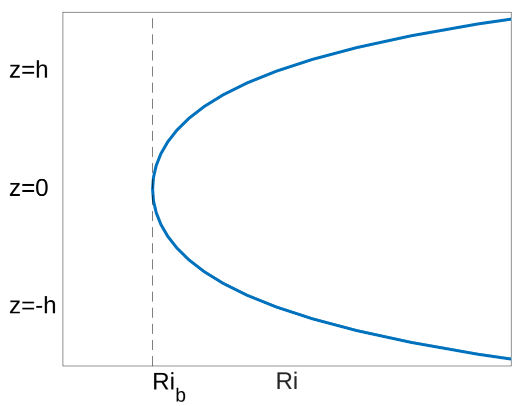


Figure 5.3: The gradient Richardson number profile for the hyperbolic tangent stratified shear layer (5.2.2). The vertical line indicates the bulk Richardson number.

- $B_z > 0$ stabilizes.

To see which effect will dominate, we must compare, but this is complicated by the fact that U_z and B_z have different dimensions: s^{-1} and s^{-2}, respectively. A direct comparison is therefore meaningless. Also, the *sign* of the shear shouldn't matter: shear drives instability regardless of its orientation. These issues are both addressed by comparing B_z with the squared shear, U_z^2. To do this, we define the gradient Richardson number:

$$Ri = \frac{B_z}{U_z^2}. \tag{5.2.1}$$

(More generally, we can write $Ri = B_z/(U_z^2 + V_z^2)$, which allows for flow in any horizontal direction.) We can imagine two limiting cases:

- $Ri \gg 1$: stratification dominates, shear is weak and we don't expect instability.
- $Ri \ll 1$: shear dominates, stratification is weak, and instability is therefore likely.

What do we expect at moderate values of Ri? Is there a critical value of Ri that separates stable and unstable regimes? The answer is yes, but the value depends on the details of the flow and various other assumptions that can be made. We'll get into this later; for now, be content to know that the critical value is typically of order unity.

As an example, consider the hyperbolic tangent model for the stably stratified shear layer:

$$U = u_0 \tanh\frac{z}{h}\,; \quad B = b_0 \tanh\frac{z}{h}. \tag{5.2.2}$$

Differentiating, we have

$$B_z = \frac{b_0}{h}\mathrm{sech}^2\frac{z}{h} \;\; ; \quad U_z^2 = \frac{u_0^2}{h^2}\mathrm{sech}^4\frac{z}{h}.$$

$$\Rightarrow \quad Ri = \frac{b_0 h}{u_0^2}\cosh^2\frac{z}{h}. \tag{5.2.3}$$

The profile is shown in figure 5.3. It is often convenient to define a bulk Richardson number, Ri_b , whose value characterizes the whole shear flow. A natural choice is the coefficient b_0h/u_0^2. In the example used here, $Ri_b = b_0h/u_0^2$ is also the minimum value of Ri.

5.3 Equilibria and perturbations

To derive the perturbation theory, we start with the Boussinesq equations for an inviscid, nondiffusive, inhomogeneous fluid. We ignore planetary rotation. The divergence equation is, as usual,

$$\vec{\nabla} \cdot \vec{u} = 0.$$

The momentum equation (1.6.5), neglecting the Coriolis acceleration and viscosity but restoring buoyancy, is

$$\frac{D\vec{u}}{Dt} = -\vec{\nabla}\pi + b\hat{e}^{(z)}, \tag{5.3.1}$$

and the buoyancy equation (1.6.9) is

$$\frac{Db}{Dt} = 0. \tag{5.3.2}$$

We assume the perturbation solution

$$\begin{aligned} \vec{u} &= U(z)\hat{e}^{(x)} + \varepsilon\vec{u}', \\ b &= B(z) + \varepsilon b', \\ \pi &= \Pi + \varepsilon\pi'. \end{aligned} \tag{5.3.3}$$

No assumption is made regarding the background pressure Π. As always, the perturbation velocity has zero divergence: $\vec{\nabla} \cdot \vec{u}' = 0$.

Substituting (5.3.3) into the momentum equation (5.3.1) gives

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + \varepsilon\vec{u}' \cdot \vec{\nabla}\right]\left[U(z,t)\hat{\imath} + \varepsilon\vec{u}'\right] = -\vec{\nabla}\left[\Pi + \varepsilon\pi'\right] + \left[B(z,t) + \varepsilon b'\right]\hat{e}^{(z)} \tag{5.3.4}$$

With no perturbation ($\varepsilon = 0$), this gives

$$\vec{\nabla}\Pi = B\hat{e}^{(z)},$$

i.e., the background pressure varies only in the vertical, where it maintains hydrostatic balance with the background buoyancy.

The $O(\varepsilon)$ terms in (5.3.4) give

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\vec{u}' + U_z w' \hat{e}^{(x)} = -\vec{\nabla}\pi' + b'\hat{e}^{(z)}. \tag{5.3.5}$$

This is the same as the homogeneous case (3.2.8) except for the second term on the right-hand side, which describes vertical accelerations due to the perturbation buoyancy.

Substitution of (5.3.3) into (5.3.2)

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + \varepsilon\vec{u}' \cdot \vec{\nabla}\right]\left[B(z) + \varepsilon b'\right] = 0. \tag{5.3.6}$$

For $\varepsilon = 0$, this gives $0 = 0$, so there is no restriction on the background buoyancy profile. The $O(\varepsilon)$ part of (5.3.6) is:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) b' + B_z w' = 0. \tag{5.3.7}$$

It is worthwhile to compare (5.3.7) with (2.2.11), the equation for buoyancy perturbations from hydrostatic equilibrium in a motionless, inhomogeneous fluid. The final term on the left hand side describes the advection of the background buoyancy gradient by the vertical velocity perturbation, just as we saw in (2.2.11). The second term on the left hand side is new; it describes the advection of buoyancy perturbations by the background flow (which was zero in the motionless case).

5.3.1 Eliminating the pressure

We eliminate the pressure, as we have done before, by combining the divergence of the momentum equation (5.3.5) with the Laplacian of its vertical component. The divergence gives a Helmholtz equation for the pressure:[1]

$$\nabla^2 \pi' = -2U_z \frac{\partial w'}{\partial x} + \frac{\partial b'}{\partial z}. \tag{5.3.8}$$

The vertical component of (5.3.5) is

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) w' = -\frac{\partial \pi'}{\partial z} + b'. \tag{5.3.9}$$

Finally, we take the Laplacian of (5.3.9) and substitute the vertical derivative of (5.3.8) to obtain:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) \nabla^2 w' - U_{zz}\frac{\partial w'}{\partial x} = \nabla_H^2 b'. \tag{5.3.10}$$

In (5.3.10) and (5.3.7), we have two equations for the two unknowns w' and b'. We substitute the normal mode forms $w' = \{\hat{w}(z)e^{\sigma t}e^{\iota(kx+\ell y)}\}_r$ and $b' = \{\hat{b}(z)e^{\sigma t}e^{\iota(kx+\ell y)}\}_r$ to obtain a pair of ordinary differential equations:

$$(\sigma + \iota kU)\nabla^2\hat{w} - \iota kU_{zz}\hat{w} = -\tilde{k}^2\hat{b} \tag{5.3.11}$$

$$(\sigma + \iota kU)\hat{b} + B_z\hat{w} = 0, \tag{5.3.12}$$

where $\nabla^2 = d^2/dz^2 - \tilde{k}^2$ as usual.

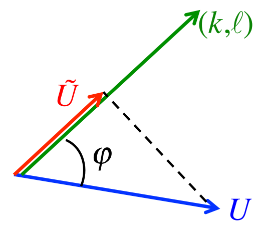
5.4 Oblique modes

A normal mode perturbation in a shear flow is affected only by the component of the background flow that is parallel to its own wave vector. To see this, inspect (5.3.11) and (5.3.12) and note that, wherever U appears, it is multiplied by k. If we define $\tilde{U}$ such that $kU = \tilde{k}\tilde{U}$ and substitute, we get a pair of equations that is isomorphic to the 2D case with wave vector $(\tilde{k}, 0)$:

$$\begin{aligned}(\sigma + \iota\tilde{k}\tilde{U})\nabla^2\hat{w} - \iota\tilde{k}\tilde{U}_{zz}\hat{w} &= -\tilde{k}^2\hat{b}\\ (\sigma + \iota\tilde{k}\tilde{U})\hat{b} + B_z\hat{w} &= 0.\end{aligned}$$

[1] Compare this with equations (2.2.14) and (3.2.12).

Figure 5.4: Definition sketch for $\tilde{U}$, the component of the background current in the direction of the wave vector.



In fact, the profile $\tilde{U}$ that we just defined is the component of U parallel to the wave vector:

$$\boxed{\tilde{U} = \frac{k}{\tilde{k}} U = U \cos \varphi,}$$

as illustrated in figure 5.4. So if the growth rate is given by a solution algorithm

$$\sigma = \mathscr{F}(z, U, B_z;\ k, l),$$

then it is also true that

$$\sigma = \mathscr{F}(z, \tilde{U}, B_z;\ \tilde{k}, 0).$$

5.5 Veering flows

In many important cases, the mean flow varies primarily with height, but is not parallel. In an Ekman spiral, for example, both speed and direction change with height. Equatorial mean currents are mostly zonal (e.g. figure **??**), but these are the exceptional. In most parts of the ocean, the mean current veers with depth.

Happily, the theory that we have already developed for parallel flows is easily extended to veering flows using our results from section 5.4 above: an instability is affected by only the component of the mean flow parallel to its own wave vector, and that component is a parallel flow.

Suppose that the mean flow of interest varies in z but has components in both horizontal directions:

$$\vec{u} = U(z)\hat{e}^{(x)} + V(z)\hat{e}^{(y)}.$$

For a given wave vector (k, ℓ), we define the parallel component of the mean flow:

$$\boxed{\tilde{U}(z) = \frac{kU(z) + \ell V(z)}{\tilde{k}}.}$$

Everything we are learning in this chapter about parallel flows is also true for veering flows if we just substitute $\tilde{U}$ for U in (5.3.11) and (5.3.12).

5.6 The Taylor-Goldstein equation

Based on the results of the previous two sections, we restrict our attention to 2D modes. Replacing $\tilde{k}$ with k (5.3.11) and (5.3.12) become

$$(\sigma + \imath kU)\left(\frac{d^2}{dz^2} - k^2\right)\hat{w} - \imath kU_{zz}\hat{w} = -k^2\hat{b} \tag{5.6.1}$$

$$(\sigma + \imath kU)\hat{b} + B_z\hat{w} = 0. \tag{5.6.2}$$

If we should need to apply these results to a 3D mode, we simply replace U by $\tilde{U}$ as defined in (5.4).

We can derive a single equation for $\hat{w}$ by solving (5.6.2) for $\hat{b}$ and substituting into (5.6.1), giving

$$(\sigma + \imath kU)\left(\frac{d^2}{dz^2} - k^2\right)\hat{w} - \imath kU_{zz}\hat{w} = k^2\frac{B_z\hat{w}}{(\sigma + \imath kU)}$$

Finally, we substitute $\sigma = -\imath kc$ and rearrange to obtain the Taylor-Goldstein (TG) equation:

$$\boxed{\hat{w}_{zz} + \left\{\frac{B_z}{(U-c)^2} + \frac{U_{zz}}{U-c} + k^2\right\}\hat{w} = 0.} \tag{5.6.3}$$

5.7 Internal waves

Solutions with real c represent waves: gravity waves, vortical waves, or some combination of the two. If we set $U = 0$ and B_z =constant in (5.6.3), we recover the dispersion relation for internal gravity waves in uniform stratification (section 2.3.1).

The limit $k \to 0$ is called the hydrostatic limit. In that limit, perturbations involve very weak vertical accelerations and are therefore in hydrostatic balance, just like the background flow. If we take this limit and also assume $U = 0$, the TG equation becomes

$$\hat{w}_{zz} + \frac{B_z}{c^2}\,\hat{w} = 0.$$

This is the equation for baroclinic normal modes, whose description may be found in any geophysical fluid dynamics text. The hydrostatic limit is a useful description not only for small-amplitude waves but also for nonlinear phenomena such as solitary waves, bores, hydraulic jumps and gravity currents.

Although we will not venture into the realm of waves here, it is important to note that the numerical methods that we are developing (sections 3.6, 7.2) apply just as well to waves as they do to instabilities. Those methods are often used to determine gravity wave and baroclinic mode characteristics in realistic situations where the stratification is not uniform and the background current is nonzero.

5.8 Kelvin-Helmholtz and Rayleigh-Taylor instabilities at an interface

Imagine an infinitely thin interface at which the velocity and the buoyancy change:

$$U = \frac{u_0}{2}\left\{\begin{array}{ll} 1, & z>0 \\ -1, & z<0 \end{array}\right. \qquad B = \frac{b_0}{2}\left\{\begin{array}{ll} 1, & z>0 \\ -1, & z<0 \end{array}\right. \tag{5.8.1}$$

As expressed in (5.6.3), the TG equation involves the second derivative U_{zz} and therefore cannot handle this discontinuity in U. To get around this problem we rephrase the TG equation in terms of the vertical displacement function η', defined by

$$w' = \frac{D\eta'}{Dt}.$$

In normal mode form this is

$$\hat{w} = (\sigma + \imath kU)\hat{\eta} = \imath k(U-c)\hat{\eta}.$$

With this change of variables (5.6.3) becomes

$$\left[(U-c)^2\hat{\eta}_z\right]_z + \left[B_z - k^2(U-c)^2\right]\hat{\eta} = 0. \tag{5.8.2}$$

(*Exercise:* check this.)

The solution is simple because U and B are constant except at the interface. Requiring that $\hat{\eta}$ be continuous and bounded for all z,

$$\hat{\eta} = \begin{cases} e^{-kz}, & z > 0 \\ e^{kz}, & z < 0 \end{cases}.$$

We pay a price for this simplicity, though. The discontinuity in $B(z)$ imposes another condition on the solution, and that condition is not so obvious. Because of that discontinuity, the derivative B_z has the form of a Dirac delta function:

$$B_z = b_0\delta(z). \tag{5.8.3}$$

Math break: the Dirac delta function
The delta function $\delta(z)$ can be thought of as a peak centered at $z = 0$, with zero width and infinite height. It has the following properties:

1. $\delta(0) = \infty$.
2. $\delta(z) = 0$, for $x \neq 0$.
3. $\int_{-\infty}^{\infty} \delta(z)dz = 1$.
4. $\int_{-\infty}^{\infty} f(z)\delta(z)dz = f(0)$.
5. $\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} f(z)\delta(z)dz = f(0)$.
6. $\lim_{\varepsilon\to 0}\int_{z_0-\varepsilon}^{z_0+\varepsilon} f(z)\delta(z-z_0)dz = f(z_0)$.

Using property 3 of the delta function, you can check that (5.8.3) integrates to give $B(z)$ as defined in (5.8.1).

We now apply the integral operation $\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} dz$ to (5.8.2). The first term gives

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\left[(U-c)^2\hat{\eta}_z\right]_z dz \;=\; \left[(U-c)^2\hat{\eta}_z\right]_{z=0-}^{z=0+} \;=\; -k\left[\left(\frac{u_0}{2}-c\right)^2 + \left(-\frac{u_0}{2}-c\right)^2\right]$$

Next,

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} B_z\hat{\eta}\, dz \;=\; \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} b_0\delta(z)\,\hat{\eta}\, dz \;=\; b_0,$$

where property 5 of the delta function has been used. The final term is

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} dz\left[k^2(U-c)^2\hat{\eta}\right] \;=\; 0.$$

This integral vanishes because the integrand is finite, so when we take the limit $\varepsilon \to 0$ the result is zero.

Combining these three integrated terms we have

$$-k\left[\left(\frac{u_0}{2}-c\right)^2 + \left(-\frac{u_0}{2}-c\right)^2\right] + b_0 = 0.$$

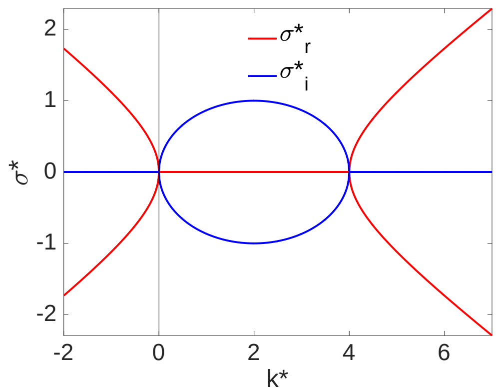


Figure 5.5: Waves and instabilities on a sharp interface at which velocity and buoyancy change discontinuously (5.8.1). The scaled wavenumber and growth rate are defined as $k^* = ku_0^2/b_0$ and $\sigma u_0/b_0$, respectively.

which we can solve for c to get

$$\frac{c}{u_0} = \pm\sqrt{\frac{b_0}{u_0^2 k} - \frac{1}{4}}.$$

or

$$\boxed{c^* = \pm\sqrt{\frac{1}{k^*} - \frac{1}{4}}\,; \quad \sigma^* \; = \; \mp\sqrt{\frac{k^{*2}}{4} - k^*}}$$

in terms of the nondimensional variables

$$c^* = \frac{c}{u_0}\,; \quad k^* = k\frac{u_0^2}{b_0}\,; \quad \sigma^* \; = \; \frac{u_0}{b_0}\sigma.$$

If the quantity under the radical is real, i.e. if

$$0 \le k^* \le 4,$$

the solution describes two waves moving oppositely (figure 5.5; blue curves). Otherwise, we have a growing and a decaying mode (red curves).

- The flow is *always* unstable, i.e. there is always a range of wavenumbers such that $k^* > 4$. These provide the simplest example of the Kelvin-Helmholtz instability: a shear instability damped by stable stratification.

- The instability exhibits ultraviolet catastrophe: the shortest waves (largest k^*) are amplified most rapidly.

- Longer waves ($0 \le k^* \le 4$) are interfacial gravity waves.

- If the buoyancy change is *unstable* ($b_0 < 0$), then the scaled wavenumber is negative (though the dimensional wavenumber is not). The interface is convectively unstable, and all disturbances are amplified. This is an example of Rayleigh-Taylor instability.

5.9 The Miles-Howard theorem

As we have discussed, the Richardson number $Ri = B_z/U_z^2$ quantifies the cometing effects of stratification and shear. If $Ri \gg 1$, stratification dominates and the flow is stable. Conversely, if $Ri \ll 1$, we expect instability. The boundary between stable and unstable flows must lie at some intermediate value of Ri which we'll call Ri_c. The Miles-Howard theorem tells us that, while Ri_c may have different values for different flow profiles, it is at most 1/4. To be precise:

> **The Miles-Howard theorem:** A necessary condition for instability in an inviscid, nondiffusive, stratified, parallel shear flow is that $Ri < 1/4$ somewhere in the flow.

To prove this theorem, we transform the TG equation via the following change of variables:

$$\hat{w} = (U-c)^{1/2}\phi.$$

The algebra is left as an exercise; the result is:

$$\left[(U-c)\phi_z\right]_z + P(z)\phi = 0, \tag{5.9.1}$$

where

$$P = \frac{B_z - \frac{1}{4}U_z^2}{U-c} - \frac{1}{2}U_{zz} - k^2(U-c).$$

We now multiply (5.9.1) by ϕ^* and integrate over the vertical domain:

$$\int_{z_B}^{z_T} \phi^*\left[(U-c)\phi_z\right]_z dz + \int_{z_B}^{z_T} \phi^* P(z)\phi\, dz = 0,$$

where z_B and z_T may be finite or infinite. Now integrate the first term by parts:

$$\phi^*(U-c)\phi_z\Big|_{z_B}^{z_T} - \int_{z_B}^{z_T} \phi_z^*(U-c)\phi_z dz + \int_{z_B}^{z_T} \phi^* P(z)\phi\, dz = 0.$$

Because $\hat{w}$ vanishes at the boundaries, the first term drops out and we have

$$-\int_{z_B}^{z_T} (U-c)|\phi_z|^2 dz + \int_{z_B}^{z_T} P(z)|\phi|^2 dz = 0.$$

The imaginary part is

$$c_i \int_{z_B}^{z_T} \left[|\phi_z|^2 + \frac{B_z - \frac{1}{4}U_z^2}{|U-c|^2}|\phi|^2 + k^2|\phi|^2\right] dz = 0.$$

The first and third terms in the integrand are positive definite. Now suppose that the second term is also positive definite. In that case, the integral is positive, and the equation can only be satisfied if $c_i = 0$. Conversely, the only way c_i can be nonzero is if the second term is negative somewhere, i.e.

$$B_z - \frac{1}{4}U_z^2 < 0\,, \quad \text{or} \;\; Ri < 1/4,$$

for some z.

5.10 Howard's semicircle theorem

The Miles-Howard theorem described in the previous section provides a condition that the mean flow must satisfy if instability is to grow. Here we'll describe a condition that the mode must satisfy.

> **Howard's semicircle theorem:** In an inviscid, stably stratified, parallel shear flow, let the background velocity $U(z)$ be bounded by U_{min} and U_{max}. Any unstable normal mode must have phase speed c located within the semicircle centered at $c_r = (U_{max} + U_{min})/2, c_i = 0$ having radius $(U_{max} - U_{min})/2$, as shown on figure 5.6.

A corollary is that the real part c_r must lie within the range of the mean flow. In other words, every unstable mode must have a critical level. This result is a generalization of the result proved in section 3.12.3 for homogeneous flows.

The proof starts off similar to that for the Miles-Howard theorem (in fact it appeared in the same paper), but is somewhat more involved. In the TG equation, we make the change of variables

$$\hat{w} = (U - c)\varphi,$$

resulting in:

$$\left[(U-c)^2\varphi_z\right]_z + \left[B_z - k^2(U-c)^2\right]\varphi = 0. \tag{5.10.1}$$

As before, we multiply by φ^* and integrate over the vertical domain. Integrating by parts, using the boundary condition $\hat{w} \to 0$, and rearranging we obtain

$$\int_{z_B}^{z_T} B_z|\phi|^2 dz = \int_{z_B}^{z_T} (U-c)^2\left[|\phi_z|^2 + k^2(U-c)^2|\phi|^2\right] dz. \tag{5.10.2}$$

The imaginary part is

$$0 = -2c_i\int_{z_B}^{z_T} (U-c_r)\left[|\phi_z|^2 + k^2|\phi|^2\right] dz. \tag{5.10.3}$$

Because the quantity in square brackets is positive definite, the integral can be zero only if $U - c_r$ changes sign. Two implications of this are worth noting:

- The discussion so far is identical to our earlier proof that an unstable mode must have a critical level in a *homogeneous* shear flow (section 3.12.3). We now see that a critical level is also necessary in *stratified* flow. The stratification term on the left-hand side of (5.10.2) becomes irrelevant when we take the imaginary part.
- In the special case $U = 0$, this result shows that c_r must be zero. This is relevant to the convective case $B_z < 0$. The convective instabilities that we explored in chapter 2 are all *stationary*. We now see that all convective instabilities must be stationary, provided only that their is no background flow.

To extend the proof, we now consider the real part of (5.10.2):

$$\int_{z_B}^{z_T} B_z|\phi|^2 dz = \int_{z_B}^{z_T} (U^2 - 2c_rU + c_r^2 - c_i^2)\left[|\phi_z|^2 + k^2|\phi|^2\right] dz. \tag{5.10.4}$$

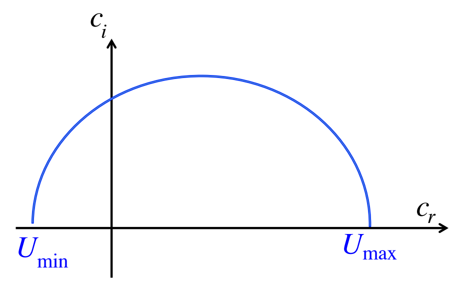


Figure 5.6: Howard's semicircle on the complex c-plane, bounded by the extremal values of the mean velocity (Howard, 1961).

The second term on the right hand side can be written as

$$-2c_r \int_{z_B}^{z_T} U\left[|\phi_z|^2 + k^2|\phi|^2\right] dz$$

which, by (5.10.3), is equal to

$$-2c_r \int_{z_B}^{z_T} c_r\left[|\phi_z|^2 + k^2|\phi|^2\right] dz.$$

Therefore, in the second term on the right-hand side of (5.10.4), we can change the U to c_r, resulting in

$$\int_{z_B}^{z_T} B_z|\phi|^2 dz = \int_{z_B}^{z_T} (U^2 - |c|^2)\left[|\phi_z|^2 + k^2|\phi|^2\right] dz. \tag{5.10.5}$$

Now if $B_z > 0$, we have

$$\int_{z_B}^{z_T} (U^2 - |c|^2)\left[|\phi_z|^2 + k^2|\phi|^2\right] dz > 0. \tag{5.10.6}$$

Now comes the cute part. Define U_{max} and U_{min} as the maximum and minimum values of $U(z)$. Note that $U_{max} - U \geq 0$ and $U_{min} - U \leq 0$, and therefore

$$(U_{max} - U)(U_{min} - U) \leq 0.$$

As a result,

$$\int_{z_B}^{z_T} (U_{max} - U)(U_{min} - U)\left[|\phi_z|^2 + k^2|\phi|^2\right] dz \leq 0,$$

or

$$\int_{z_B}^{z_T} \left[U_{max}U_{min} - U(U_{max} + U_{min}) + U^2\right]\left[|\phi_z|^2 + k^2|\phi|^2\right] dz \leq 0. \tag{5.10.7}$$

Note that the integrand is the product of the two factors in square brackets. We'll now convert the first of these factors to a constant, which we can then remove from the integral. We do this in two steps. First, as we noted above, (5.10.3) allows us to replace U with c_r in the second term. Now consider the third term, which contains U^2. By (5.10.6),

$$\int_{z_B}^{z_T} |c|^2\left[|\phi_z|^2 + k^2|\phi|^2\right] dz < \int_{z_B}^{z_T} U^2\left[|\phi_z|^2 + k^2|\phi|^2\right] dz.$$

So if we replace U^2 by $|c|^2$ in (5.10.7), the inequality is still true:

$$\int_{z_B}^{z_T} \left[U_{max}U_{min} - c_r(U_{max} + U_{min}) + |c|^2\right]\left[|\phi_z|^2 + k^2|\phi|^2\right] dz \leq 0. \tag{5.10.8}$$

	$BP < 0$	$BP > 0$
$SP < 0$	stable	convective instability opposed by shear
$SP > 0$	shear instability opposed by buoyancy	sheared convection

Table 5.1: Categorizing instabilities by their energy source.

Given that the first factor in the integrand is a constant and the second is positive definite, the inequality can be true only if the first factor is negative:

$$U_{max}U_{min} - c_r(U_{max} + U_{min}) + |c|^2 \leq 0.$$

After some juggling, this becomes

$$\left(c_r - \frac{U_{max} + U_{min}}{2}\right)^2 + c_i^2 \leq \left(\frac{U_{max} - U_{min}}{2}\right)^2. \tag{5.10.9}$$

This inequality describes the interior of a circle on the complex c plane whose radius is $(U_{max} - U_{min})/2$ and whose center is on the real axis at $(U_{max} + U_{min})/2$. For any unstable mode, $c_i > 0$ and therefore c must lie in the upper half of the circle, i.e. Howard's semicircle as shown in figure 5.6.

5.11 The vertical buoyancy flux

To analyze the perturbation kinetic energy, we repeat the analysis of section 3.9 beginning with the momentum equation (5.3.5) instead of (3.2.8). The only difference is the vertical buoyancy term $b'\hat{e}^{(z)}$. Therefore, converting to normal mode form gives the same results for the continuity and horizontal momentum equations, (6.10.1-6.10.3), but the vertical momentum equation contains the normal mode for buoyancy:

$$(\sigma + \imath kU)\,\hat{w} = -\hat{\pi}_z + \hat{b} + \nu\nabla^2\hat{w}. \tag{5.11.1}$$

As in section 6.10, we multiply the momentum equations by the conjugates of the velocity eigenfunctions, take the real part and divide by 2. The result is just (6.10.8) with an added term:

$$\boxed{2\sigma_r K = SP - EF_z + BF + \nu K_{zz} - \varepsilon,} \tag{5.11.2}$$

where

$$BF = \overline{w'b'} = \frac{1}{2}(\hat{w}^*\hat{b})_r$$

is the buoyancy flux, also called the buoyancy production. When buoyant fluid rises and dense fluid sinks, $BF > 0$. Therefore, BF is the second term we have found (after SP) that is capable of creating kinetic energy[2]. This term is the source of kinetic energy for convective instability. In contrast $BF < 0$ means that the instability has to do work against gravity in order to grow, i.e. it must lift dense fluid and depress buoyant fluid. The latter case is more common in stable stratification ($B_z > 0$). It is useful to categorize instabilities according to their energy source (table 5.1).

[2]More precisely, it converts potential energy into kinetic energy.

5.12 Summary

- The gradient Richardson number $Ri = B_z/U_z^2$ compares the damping effect of stable stratification and the destabilizing effect of shear.
- The Taylor-Goldstein equation (5.6.3) describes a wide array of instabilities and wavelike phenomena in the inviscid limit.
- A normal mode disturbance in a shear flow is affected only by that component of the background flow that is parallel to its own wave vector.
- The Miles-Howard theorem states that a stratified, inviscid, nondiffusive parallel shear flow can be unstable only if $Ri(z) < 1/4$ for some z.
- The Howard semicircle theorem states that the complex phase speed of a growing mode must lie within the semicircle shown in figure 5.6.

5.13 Further reading

Howard, Louis N., 1961: "Note on a paper of John W. Miles." *Journal of Fluid Mechanics*, **10**: 509-512.

Miles, John W., 1961: "On the stability of heterogeneous shear flows." *Journal of Fluid Mechanics*, **10**: 496-508.

Smyth, W.D. and J.N. Moum, 2012: "Ocean mixing by Kelvin-Helmholtz instability", *Oceanography* **5(2)**, 140-149.